Generalized Finsler Geometry in Einstein, String and Metric–Affine Gravity

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Abstract

We develop the method of anholonomic frames with associated nonlinear connection (in brief, N–connection) structure and show explicitly how geometries with local anisotropy (various type of Finsler–Lagrange–Cartan–Hamilton geometry) can be modeled in the metric–affine spaces. There are formulated the criteria when such generalized Finsler metrics are effectively induced in the Einstein, teleparallel, Riemann–Cartan and metric–affine gravity. We argue that every generic off–diagonal metric (which can not be diagonalized by coordinate transforms) is related to specific N–connection configurations. We elaborate the concept of generalized Finsler–affine geometry for spaces provided with arbitrary N–connection, metric and linear connection structures and characterized by gravitational field strengths, i. e. by nontrivial N–connection curvature, Riemannian curvature, torsion and nonmetricity. We apply a irreducible decomposition techniques (in our case with additional N–connection splitting) and study the dynamics of metric–affine gravity fields generating Finsler like configurations. The classification of basic eleven classes of metric–affine spaces with generic local anisotropy is presented.

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Contents

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1 Introduction

Brane worlds and related string and gauge theories define the paradigm of modern physics and have generated enormous interest in higher–dimensional spacetimes amongst particle and astrophysics theorists (see recent advances in Refs. [1, 2, 3] and an outline of the gauge idea and gravity in Refs. [4, 5]). The unification scheme in the framework of string/ brane theory indicates that the classical (psedo) Riemannian description is not valid on all scales of interactions. It turns out that low–energy dilaton and axi–dilaton interactions are tractable in terms of non–Riemannian mathematical structures possessing in particular anholonomic (super) frame [equivalently, (super) vielbein] fields [6], noncommutative geometry [7], quantum group structures [8] all containing, in general, nontrivial torsion and nonmetricity fields. For instance, in the closest alternatives to general relativity theory, the teleparallel gravity models [9], the spacetime is of Witzenbock type with trivial curvature but nontrivial torsion. The frame or coframe filed (tetrad, vierbein, in four dimensions, 4D) is the basic dynamical variable treated as the gauge potential corresponding to the group of local translations.

Nowadays, it was established a standard point of view that a number of low energy (super) string and particle physics interactions, at least the nongravitational ones, are described by (super) gauge potentials interpreted as linear connections in suitable (super) bundle spaces. The formal identity between the geometry of fiber bundles [10] is recognized since the works [11] (see a recent discussing in connection to a unified description in of interactions in terms of composite fiber bundles in Ref. [12]).

The geometry of fiber bundles and the moving frame method originating from the E. Cartan works [6] constitute a modern approach to Finsler geometry and generalizations (also suggested by E. Cartan [13] but finally elaborated in R. Miron and M. Anastasiei works [14]), see some earlier and recent developments in Refs. [15, 16, 17, 18, 19]. Various type of geometries with local anisotropy (Finsler, Lagrange, Hamilton, Cartan and their generalizations, according to the terminology proposed in [14]), are modeled on (co) vector / tangent bundles and their higher order generalizations [21, 20] with different applications in Lagrange and Hamilton mechanics or in generalized Finsler gravity. Such constructions were defined in low energy limits of (super) string theory and supergravity [22, 23] and generalized for spinor bundles [24] and affine– de Sitter frame bundles [25] provided with nonlinear connection (in brief, N–connection) structure of first and higher order anisotropy.

The gauge and moving frame geometric background is also presented in the metric–affine gravity (MAG) [4]. The geometry of this theory is very general being described by the two–forms of curvature and of torsion and the one–form of nonmetricity treated respectively as the gravitational field strengths for the linear connection, coframe and metric. The kinematic scheme of MAG is well understood at present time as well certain dynamical aspects of the vacuum configurations when the theory can be reduced to an effective Einstein–Proca model with nontrivial torsion and nonmetricity [26, 27, 28, 29]. There were constructed a number of exact solutions in MAG connecting the theory to modern string gravity and another extra dimension generalizations [30, 31, 32]. Nevertheless, one very important aspect has not been yet considered. As a gauge theory, the MAG can be expressed with respect to arbitrary frames and/or coframes. So, if we introduce frames with associated N–connection structure, the MAG should incorporate models with generic local anisotropy

(Finsler like ones and their generalizations) which are distinguished by certain prescriptions for anholonomic frame transforms, N–connection coefficients and metric and linear connection structures adapted to such anholonomic configurations. Roughly speaking, the MAG contains the bulk of known generalized Finsler geometries which can be modeled on metric–affine spaces by defining splittings on subspaces like on (co) vector/ tangent bundles and considering certain anholonomically constrained moving frame dynamics and associated N–connection geometry.

Such metric–affine spaces with local anisotropy are enabled with generic off–diagonal metrics which can not be diagonalized by any coordinate transforms. The off–diagonal coefficients can be mapped into the components of a specific class of anholonomic frames, defining also the coefficients of the N–connection structure. It is possible to redefine equivalently all geometrical values like tensors, spinors and connections with respect to such N–adapted anholonomic bases. If the N–connection, metric and linear connections are chosen for an explicit type of Finsler geometry, a such geometric structure is modeled on a metric–affine space (we claim that a Finsler–affine geometry is constructed). The point is to find explicitly by what type of frames and connections a locally anisotropic structure can be modeled by exact solutions in the framework of MAG. Such constructions can be performed in the Einstein–Proca sector of the MAG gravity and they can be defined even in general relativity theory (see the partners of this paper with field equations and exact solutions in MAG modeling Finsler like metrics and generalizations [33]).

Within the framework of moving frame method [6], we investigated in a series of works [34, 35, 36, 37] the conditions when various type of metrics with noncommutative symmetry and/or local anisotropy can be effectively modeled by anholonomic frames on (pseudo) Riemannian and Riemann–Cartan spaces [38]. We constructed explicit classes of such exact solutions in general relativity theory and extra dimension gravity models. They are parametrized by generic off–diagonal metrics which can not diagonalized by any coordinate transforms but only by anholonomic frame transforms. The new classes of solutions describe static black ellipsoid objects, locally anistoropic configurations with toroidal and/or ellipsoidal symmetries, wormholes/ flux tubes and Taub-NUT metrics with polarizied constants and various warped spinor–soliton–dilaton configurations. For certain conditions, some classes of such solutions preserve the four dimensional (4D) local Lorentz symmetry but, in general, they are with violated Lorentz symmetry in the bulk.

Our ongoing effort is to model different classes of geometries following a general approach to the geometry of (co) vector/tangent bundles and affine–de Sitter frame bundles [25] and superbundles [23] and or anisotropic spinor spaces [24] provided with N–connection stuctures. The basic geometric objects on such spaces are defined by proper classes of anholonomic frames and associated N–connections and correspondingly adapted metric and linear connections. There are examples when certain Finsler like configurations are modeled even by some exact solutions in Einstein or Einstein–Cartan gravity and, inversely (the outgoing effort), by using the almost Hermitian formulation [14, 20, 24] of Lagrange/Hamilton and Finsler/Cartan geometry, we can consider Einstein and gauge gravity models defined on tangen/cotangent and vector/covector bundles. Recently, there were also obtained some explicit results demonstrating that the anholonomic frames geometry has a natural connection to noncommutative geometry in string/M–theory and noncommutative gauge models of gravity [36, 37] (on existing approaches to noncommutative geometry and gravity we cite

Refs. [7]).

We consider torsion fields induced by anholonomic vielbein transforms when the theory can be extended to a gauge [5], metric–affine [4], a more particular Riemann–Cartan case [38], or to string gravity with B–field [2]. We are also interested to define the conditions when an exact solution possesses hidden noncommutative symmetries, induced torsion and/or locally anisotropic configurations constructed, for instance, in the framework of the Einstein theory. This direction of investigation develops the results obtained in Refs. [35] and should be distinguished from our previous works on the geometry of Clifford and spinor structures in generalized Finsler and Lagrange–Hamilton spacetimes [24]. Here we emphasize that the works [34, 35, 36, 37, 24] were elaborated following general methods of the geometry of anholonomic frames with associated N–connections in vector (super) bundles [14, 20, 23]. The concept of N–connection was proposed in Finsler geometry [15, 17, 18, 19, 16, 13]. As a set of coefficients it was firstly present the E. Cartan's monograph[13] and then was elaborated in a more explicit form by A. Kawaguchi [39]. It was proven that the N–connection structures can be defined also on (pseudo) Riemannian spaces and certain methods work effectively in constructing exact solutions in Einstein gravity [24, 34, 35].

In order to avoid possible terminology ambiguities, we note that for us the definition of N–connection is that proposed in global form by W. Barthel in 1963 [40] when a N–connection is defined as an exact sequence related to a corresponding Whitney sum of the vertical and horizontal subbundles, for instance, in a tangent vector bundle. [1] This concept is different from that accepted in Ref. [42] were the term 'nonlinear connection' is used for tetrads as N–connections which do not transform inhomogeneously under local frame rotations. That approach invokes nonlinear realizations of the local spacetime group (see also an early model of gauge gravity with nonlinear gauge group realizations [43] and its extensions to Finsler like [25] or noncommutative gauge gravity theories [36]).

In summary, the aim of the present work is to develop a unified scheme of anholonomic frames with associated N–connection structure for a large number of gauge and gravity models (in general, with locally isotropic and anisotropic interactions and various torsion and nonmetricity contributions) and effective generalized Finsler–Weyl–Riemann–Cartan geometries derived from MAG. We elaborate a detailed classification of such spaces with nontrivial N–connection geometry. The unified scheme and classification were inspired by a number of exact solutions parametrized by generic off–diagonal metrics and anholonomic frames in Einstein, Einstein–Cartan and string gravity. The resulting formalism admits inclusion of locally anisotropic spinor interactions and extensions to noncommutative geometry and string/brane gravity [22, 23, 34, 35, 36, 37]. Thus, the geometry of metric–affine spaces enabled with an additional N–connection structure is sufficient not only to model the bulk of physically important non–Riemannian geometries on (pseudo) Riemannian spaces but also states the conditions when effective spaces with generic anisotropy can be derived as exact solutions of gravitational and matter field equations. In the present work we pay attention to the geometrical (pre–dynamical) aspects of the generalized Finsler–affine grav-

[1] Instead of a vector bundle we can consider a tangent bundle, or cotangent/covector ones, or even general manifolds of necessary smooth class with adapted definitions of global sums of horizonal and vertical subspaces. The geometry of N–connections is investigated in details in Refs. [41, 14, 23, 24, 25] for various type of spaces.

ity which constitute a theoretical background for constructing a number of exact solutions in MAG in the partner papers [33].

The article is organized as follows. We begin, in Sec. 2, with a review of the main concepts from the metric–affine geometry and the geometry of anholonomic frames with associated N–connections. We introduce the basic definitions and formulate and prove the main theorems for the N–connection, linear connection and metric structures on metric–affine spaces and derive the formulas for torsion and curvature distinguished by N–connections. Next, in Sec. 3, we state the main properties of the linear and nonlinear connections modeling Finsler spaces and their generalizations and consider how the N–connection structure can be derived from a generic off–diagonal metric in a metric–affine space. Section 4 is devoted to the definition and investigation of generalized Finsler–affine spaces. We illustrate how by corresponding parametrizations of the off–diagonal metrics, anholonomic frames, N–connections and distinguished connections every type of generalized Finsler–Lagrange–Cartan–Hamilton geometry can be modeled in the metric–affine gravity or any its restrictions to the Einstein–Cartan and general relativity theory. In Sec. 5, we conclude the results and point out how the synthesis of the Einstein, MAG and generalized Finsler gravity models can be realized and connected to the modern string gravity. In Appendix we elaborate a detailed classification of eleven classes of spaces with generic local anisotropy (i. e. possessing nontrivial N–connection structure) and various types of curvature, torsion and nonmetricity distinguished by N–connections.

Our basic notations and conventions combine those from Refs. [4, 14, 34, 35] and contain an interference of traditions from MAG and generalized Finsler geometry. The spacetime is modeled as a manifold V^{n+m} of necessary smoothly class of dimension $n+m$. The Greek indices $\alpha, \beta, ...$ can split into subclasses like $\alpha = (i, a)$, $\beta = (j, b)$ ... where the Latin indices from the middle of the alphabet, $i, j, k, ...$ run values $1, 2, ...n$ and the Latin indices from the beginning of the alphabet, $a, b, c, ...$ run values $n+1, n+2$, ..., $n+m$. We follow the Penrose convention on abstract indices [44] and use underlined indices like $\underline{\alpha} = (\underline{i}, \underline{a})$, for decompositions with respect to coordinate frames. The notations for connections $\Gamma^{\alpha}{}_{\beta\gamma}$, metrics $g_{\alpha\beta}$ and frames e_α and coframes ϑ^β, or other geometrical and physical objects, are the standard ones from MAG if a nonlinear connection (N–connection) structure is not emphasized on the spacetime. If a N–connection and corresponding anholonomic frame structure are prescribed, we use "boldfaced" symbols with possible splitting of the objects and indices like $\mathbf{V}^{n+m}$, $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$, $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, $\mathbf{e}_\alpha = (e_i, e_a)$, ...being distinguished by N–connection (in brief, we use the terms d–objects, d–tensor, d–connection in order to say that they are for a metric–affine space modeling a generalized Finsler, or another type, anholonomic frame geometry). The symbol "$\doteqdot$" will be used is some formulas which state that the relation is introduced "by definition" and the end of proofs will be stated by symbol $\blacksquare$.

2 Metric–Affine Spaces and Nonlinear Connections

We outline the geometry of anholonomic frames and associated nonlinear connections (in brief, N–connections) in metric–affine spaces which in this work are necessary smooth class manifolds, or (co) vector/ tangent bundles provided with, in general, indepen-

dent nonlinear and linear connections and metrics, and correspondingly derived strengths like N–connection curvature, Riemannian curvature, torsion and nonmetricity. The geometric formalism will be applied in the next Sections where we shall prove that every class of (pseudo) Riemannian, Kaluza–Klein, Einstein–Cartan, metric–affine and generalized Lagrange–Finsler and Hamilton–Cartan spaces is characterized by corresponding N–connection, metric and linear connection structures.

2.1 Linear connections, metrics and anholonomic frames

We briefly review the standard results on linear connections and metrics (and related formulas for torsions, curvatures, Ricci and Einstein tensors and Bianchi identities) defined with respect to arbitrary anholonomic bases in order to fix a necessary reference which will be compared with generalized Finsler–affine structures we are going to propose in the next sections for spaces provided with N–connection. The results are outlined in a form with conventional splitting into horizontal and vertical subspaces and subindices. We follow the Ref. [45] but we use Greek indices and denote a covariant derivative by D preserving the symbol $\bigtriangledown$ for the Levi–Civita (metric and torsionless) connection. Similar formulas can be found, for instance. in Ref. [46].

Let V^{n+m} be a $(n+m)$–dimensional underlying manifold of necessary smooth class and denote by TV^{n+m} the corresponding tangent bundle. The local coordinates on $V^{n+m}, u = \{u^{\underline{\alpha}} = (x^{\underline{i}}, y^{\underline{a}})\}$ conventionally split into two respective subgroups of "horizontal" coordinates (in brief, h–coordinates), $x = (x^{\underline{i}})$, and "vertical" coordinates (v–coordinates), $y = (y^{\underline{a}})$, with respective indices running the values $\underline{i}, \underline{j}, ... = 1, 2, ..., n$ and $\underline{a}, \underline{b}, ... = n+1, n+2, ..., n+m$. The splitting of coordinates is treated as a formal labeling if any fiber and/or the N–connection structures are not defined. Such a splitting of abstract coordinates $u^{\alpha} = (x^i, y^a)$ may be considered, for instance, for a general (pseudo) Riemannian manifold with x^i being some 'holonomic" variables (unconstrained) and y^a being "anholonomic" variables (subjected to some constraints), or in order to parametrize locally a vector bundle (E, μ, F, M) defined by an injective surjection $\mu : E \rightarrow M$ from the total space E to the base space M of dimension $\dim M = n$, with F being the typical vector space of dimension $\dim F = m$. For our purposes, we consider that both M and F can be, in general, provided with metric structures of arbitrary signatures. On vector bundles, the values $x = (x^i)$ are coordinates on the base and $y = (y^a)$ are coordinates in the fiber. If $\dim M = \dim F$, the vector bundle E transforms into the tangent bundle TM. The same conventional coordinate notation $u^{\alpha} = (x^i, y^a \rightarrow p_a)$ can be used for a dual vector bundle (E, μ, F^*, M) with the typical fiber F^* being a covector space (of 1-forms) dual to F, where p_a are local (dual) coordinates. For simplicity, we shall label y^a as general coordinates even for dual spaces if this will not result in ambiguities. In general, our geometric constructions will be elaborated for a manifold V^{n+m} (a general metric–affine spaces) with some additional geometric structures and fibrations to be stated or modeled latter (for generalized Finsler geometries) on spacetimes under consideration.

At each point $p \in V^{n+m}$, there are defined basis vectors (local frames, vielbeins) $e_{\alpha} = A_{\alpha}^{\ \underline{\alpha}}(u)\partial_{\underline{\alpha}} \in TV^{n+m}$, with $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ being tangent vectors to the local coordinate lines $u^{\underline{\alpha}} = u^{\underline{\alpha}}(\tau)$ with parameter τ. In every point p, there is also a dual basis $\vartheta^{\beta} = A^{\beta}_{\ \underline{\beta}}(u)du^{\underline{\beta}}$ with $du^{\underline{\beta}}$ considered as coordinate one forms. The duality conditions can be written in

abstract form by using the interior product $\rfloor$, $e_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha$, or in coordinate form $A_\alpha^{\ \underline{\alpha}} A^\beta_{\ \underline{\alpha}} = \delta^\beta_\alpha$, where the Einstein rule of summation on index $\underline{\alpha}$ is considered, δ^β_α is the Kronecker symbol. The "not underlined" indices $\alpha, \beta, ...$, or $i, j, ...$ and $a, b, ...$ are treated as abstract labels (as suggested by R. Penrose). We shall underline the coordinate indices only in the cases when it will be necessary to distinguish them from the abstract ones.

Any vector and 1–form fields, for instance, X and, respectively, $\widetilde{Y}$ on V^{n+m} are decomposed in h– and v–irreducible components,

$$X = X^\alpha e_\alpha = X^i e_i + X^a e_a = X^{\underline{\alpha}} \partial_{\underline{\alpha}} = X^{\underline{i}} \partial_{\underline{i}} + X^{\underline{a}} \partial_{\underline{a}}$$

and

$$\widetilde{Y} = \widetilde{Y}_\alpha \vartheta^\alpha = \widetilde{Y}_i \vartheta^i + \widetilde{Y}_a \vartheta^a = \widetilde{Y}_{\underline{\alpha}} du^{\underline{\alpha}} = \widetilde{Y}_{\underline{i}} dx^{\underline{i}} + \widetilde{Y}_{\underline{a}} dy^{\underline{a}}.$$

We shall omit labels like "$\sim$" for forms if this will not result in ambiguities.

Definition 2.1. *A linear (affine) connection D on V^{n+m} is a linear map (operator) sending every pair of smooth vector fields (X, Y) to a vector field $D_X Y$ such that*

$$D_X (sY + Z) = s D_X Y + D_X Z$$

for any scalar $s = const$ and for any scalar function $f(u^\alpha)$,

$$D_X (fY) = f D_X Y + (Xf) Y \text{ and } D_X f = Xf.$$

$D_X Y$ is called the covariant derivative of Y with respect to X (this is not a tensor). But we can always define a tensor $DY : X \to D_X Y$. The value DY is a $(1,1)$ tensor field and called the covariant derivative of Y.

With respect to a local basis e_α, we can define the scalars $\Gamma^\alpha_{\ \beta\gamma}$, called the components of the linear connection D, such that

$$D_\alpha e_\beta = \Gamma^\gamma_{\ \beta\alpha} e_\gamma \text{ and } D_\alpha \vartheta^\beta = -\Gamma^\beta_{\ \gamma\alpha} \vartheta^\gamma$$

were, by definition, $D_\alpha \doteqdot D_{e_\alpha}$ and because $e_\beta \vartheta^\beta = const$.

We can decompose

$$D_X Y = (D_X Y)^\beta e_\beta = \left[e_\alpha (Y^\beta) + \Gamma^\beta_{\ \gamma\alpha} \vartheta^\gamma \right] e_\beta \doteqdot Y^\beta_{\ ;\alpha} X^\alpha \qquad (1)$$

where $Y^\beta_{\ ;\alpha}$ are the components of the tensor DY.

It is a trivial proof that any change of basis (vielbein transform), $e_{\alpha'} = B_{\alpha'}^{\ \alpha} e_\alpha$, with inverse $B^{\alpha'}_{\ \alpha}$, results in a corresponding (nontensor) rule of transformation of the components of the linear connection,

$$\Gamma^{\alpha'}_{\ \beta'\gamma'} = B^{\alpha'}_{\ \alpha} \left[B_{\beta'}^{\ \beta} B_{\gamma'}^{\ \gamma} \Gamma^\alpha_{\ \beta\gamma} + B_{\gamma'}^{\ \gamma} e_\gamma \left(B_{\beta'}^{\ \alpha} \right) \right]. \qquad (2)$$

Definition 2.2. *A local basis e_β is anhlonomic (nonholonomic) if there are satisfied the conditions*

$$e_\alpha e_\beta - e_\beta e_\alpha = w^\gamma_{\alpha\beta} e_\gamma \qquad (3)$$

for certain nontrivial anholonomy coefficients $w^\gamma_{\alpha\beta} = w^\gamma_{\alpha\beta}(u^\tau)$. A such basis is holonomic if $w^\gamma_{\alpha\beta} \doteqdot 0$.

For instance, any coordinate basis ∂_α is holonomic. Any holonomic basis can be transformed into a coordinate one by certain coordinate transforms.

Definition 2.3. *The torsion tensor is a tensor field $\mathcal{T}$ defined by*

$$\mathcal{T}(X,Y) = D_X Y - D_Y X - [X,Y], \tag{4}$$

where $[X,Y] = XY - YX$, for any smooth vector fields X and Y.

The components $T^{\gamma}{}_{\alpha\beta}$ of a torsion $\mathcal{T}$ with respect to a basis e_α are computed by introducing $X = e_\alpha$ and $Y = e_\beta$ in (4),

$$\mathcal{T}\ (e_\alpha, e_\beta) = D_\alpha e_\beta - D_\beta e_\alpha - [e_\alpha, e_\beta] = T^{\gamma}{}_{\alpha\beta} e_\gamma$$

where

$$T^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\beta\alpha} - \Gamma^{\gamma}{}_{\alpha\beta} - w^{\gamma}_{\alpha\beta}. \tag{5}$$

We note that with respect to anholonomic frames the coefficients of anholonomy $w^{\gamma}_{\alpha\beta}$ are contained in the formula for the torsion coefficients (so any anholonomy induces a specific torsion).

Definition 2.4. *The Riemann curvature tensor $\mathcal{R}$ is defined as a tensor field*

$$\mathcal{R}(X,Y)\,Z = D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z. \tag{6}$$

We can compute the components $R^{\alpha}{}_{\beta\gamma\tau}$of curvature $\mathcal{R}$, with respect to a basis e_α are computed by introducing $X = e_\gamma, Y = e_\tau$, $Z = e_\beta$ in (6). One obtains

$$\mathcal{R}(e_\gamma, e_\tau)\, e_\beta = R^{\alpha}{}_{\beta\gamma\tau} e_\alpha$$

where

$$R^{\alpha}{}_{\beta\gamma\tau} = e_\tau\left(\Gamma^{\alpha}{}_{\beta\gamma}\right) - e_\gamma\left(\Gamma^{\alpha}{}_{\beta\tau}\right) + \Gamma^{\nu}{}_{\beta\gamma}\Gamma^{\alpha}{}_{\nu\tau} - \Gamma^{\nu}{}_{\beta\tau}\Gamma^{\alpha}{}_{\nu\gamma} + w^{\nu}_{\gamma\tau}\Gamma^{\alpha}{}_{\beta\nu}. \tag{7}$$

We emphasize that the anholonomy and vielbein coefficients are contained in the formula for the curvature components (6). With respect to coordinate frames, $e_\tau = \partial_\tau$, with $w^{\nu}_{\gamma\tau} = 0$, we have the usual coordinate formula.

Definition 2.5. *The Ricci tensor $\mathcal{R}i$ is a tensor field obtained by contracting the Riemann tensor,*

$$R_{\ \beta\tau} = R^{\alpha}{}_{\beta\tau\alpha}. \tag{8}$$

We note that for a general affine (linear) connection the Ricci tensor is not symmetric $R_{\ \beta\tau} \doteqdot R_{\ \tau\beta}$.

Definition 2.6. *A metric tensor is a $(0,2)$ symmetric tensor field*

$$g = g_{\alpha\beta}(u^\gamma)\vartheta^\alpha \otimes \vartheta^\beta$$

defining the quadratic (length) linear element,

$$ds^2 = g_{\alpha\beta}(u^\gamma)\vartheta^\alpha\vartheta^\beta = g_{\underline{\alpha}\underline{\beta}}(u^{\underline{\gamma}})du^{\underline{\alpha}}du^{\underline{\beta}}.$$

For physical applications, we consider spaces with local Minkowski singnature, when locally, in a point u_0^{γ}, the diagonalized metric is $g_{\underline{\alpha}\underline{\beta}}(u_0^{\gamma}) = \eta_{\underline{\alpha}\underline{\beta}} = (1,-1,-1,...)$ or, for our further convenience, we shall use metrics with the local diagonal ansatz being defined by any permutation of this order.

Theorem 2.1. *If a manifold V^{n+m} is enabled with a metric structure g, then there is a unique torsionless connection, the Levi–Civita connection $D = \bigtriangledown$, satisfying the metricity condition*

$$\bigtriangledown g = 0. \tag{9}$$

The proof, as an explicit construction, is given in Ref. [45]. Here we present the formulas for the components $\Gamma^{\alpha}_{\bigtriangledown\ \beta\tau}$ of the connection $\bigtriangledown$, computed with respect to a basis e_τ,

$$\begin{aligned} \Gamma_{\bigtriangledown\ \alpha\beta\gamma} &= g\left(e_\alpha, \bigtriangledown_\gamma e_\beta\right) = g_{\alpha\tau}\Gamma^{\tau}_{\bigtriangledown\ \alpha\beta} \\ &= \frac{1}{2}\left[e_\beta\left(g_{\alpha\gamma}\right) + e_\gamma\left(g_{\beta\alpha}\right) - e_\alpha\left(g_{\gamma\beta}\right) + w^{\tau}{}_{\gamma\beta}g_{\alpha\tau} + w^{\tau}{}_{\alpha\gamma}g_{\beta\tau} - w^{\tau}{}_{\beta\gamma}g_{\alpha\tau}\right]. \end{aligned} \tag{10}$$

By straightforward calculations , we can check that

$$\bigtriangledown_\alpha g_{\beta\gamma} = e_\alpha\left(g_{\beta_\gamma}\right) - \Gamma^{\tau}_{\bigtriangledown\ \beta\alpha}g_{\tau_\gamma} - \Gamma^{\tau}_{\bigtriangledown\ \gamma\alpha}g_{\beta\tau} \equiv 0$$

and, using the formula (5),

$$T^{\gamma}_{\bigtriangledown\ \alpha\beta} = \Gamma^{\gamma}_{\bigtriangledown\ \beta\alpha} - \Gamma^{\gamma}{}_{\bigtriangledown\alpha\beta} - w^{\gamma}_{\alpha\beta} \equiv 0.$$

We emphasize that the vielbein and anholonomy coefficients are contained in the formulas for the components of the Levi–Civita connection $\Gamma^{\tau}_{\bigtriangledown\ \alpha\beta}$ (10) given with respect to an anholonomic basis e_α. The torsion of this connection, by definition, vanishes with respect to all bases, anholonomic or holonomic ones. With respect to a coordinate base ∂_α, the components $\Gamma_{\bigtriangledown\ \alpha\beta\gamma}$ (10) transforms into the so–called 1-st type Christoffel symbols

$$\Gamma^{\bigtriangledown}_{\alpha\beta\gamma} = \Gamma^{\{\}}_{\alpha\beta\gamma} = \{\alpha\beta\gamma\} = \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\gamma g_{\beta\alpha} - \partial_\alpha g_{\gamma\beta}\right). \tag{11}$$

If a space V^{n+m} posses a metric tensor, we can use $g_{\alpha\beta}$ and the inverse values $g^{\alpha\beta}$ for lowering and upping indices as well to contract tensor objects.

Definition 2.7.
a) The Ricci scalar R is defined

$$R \doteqdot g^{\alpha\beta}R_{\alpha\beta},$$

where $R_{\alpha\beta}$ is the Ricci tensor (8).
b) The Einstein tensor $\mathcal{G}$ has the coefficients

$$G_{\alpha\beta} \doteqdot R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta},$$

with respect to any anholonomic or anholonomic frame e_α.

We note that $G_{\alpha\beta}$ and $R_{\alpha\beta}$ are symmetric only for the Levi–Civita connection $\bigtriangledown$ and that $\bigtriangledown_\alpha G^{\alpha\beta}=0.$

It should be emphasized that for any general affine connection D and metric g structures the metric compatibility conditions (9) are not satisfied.

Definition 2.8. *The nonmetricity field*

$$\mathcal{Q}=Q_{\alpha\beta}\ \vartheta^\alpha\otimes\vartheta^\beta$$

on a space V^{n+m} is defined by a tensor field with the coefficients

$$Q_{\gamma\alpha\beta}\doteqdot -D_\gamma g_{\alpha\beta} \tag{12}$$

where the covariant derivative D is defined by a linear connection 1–form $\Gamma^{\gamma}{}_{\alpha}=\Gamma^{\gamma}{}_{\alpha\beta}\vartheta^\beta.$

In result, we can generalize the concept of (pseudo) Riemann space [defined only by a locally (pseudo) Euclidean metric inducing the Levi–Civita connection with vanishing torsion] and Riemann–Cartan space [defined by any independent metric and linear connection with nontrivial torsion but with vanishing nonmetricity] (see details in Refs. [4, 38]):

Definition 2.9. *A metric–affine space is a manifold of necessary smooth class provided with independent linear connection and metric structures. In general, such spaces posses nontrivial curvature, torsion and nonmetricity (called strength fields).*

We can extend the geometric formalism in order to include into consideration the Finsler spaces and their generalizations. This is possible by introducing an additional fundamental geometric object called the N–connection.

2.2 Anholonomic frames and associated N–connections

Let us define the concept of nonlinear connection on a manifold V^{n+m}. [2] We denote by $\pi^T: TV^{n+m}\to TV^n$ the differential of the map $\pi: V^{n+m}\to V^n$ defined as a fiber–preserving morphism of the tangent bundle (TV^{n+m},τ_E,V^n) to V^{n+m} and of tangent bundle (TV^n,τ,V^n). The kernel of the morphism π^T is a vector subbundle of the vector bundle $(TV^{n+m},\tau_E,V^{n+m})$. This kernel is denoted $(vV^{n+m},\tau_V,V^{n+m})$ and called the vertical subbundle over V^{n+m}. We denote the inclusion mapping by $i: vV^{n+m}\to TV^{n+m}$ when the local coordinates of a point $u\in V^{n+m}$ are written $u^\alpha=(x^i,y^a)$, where the values of indices are $i,j,k,...=1,2,...,n$ and $a,b,c,...=n+1,n+2,...,n+m.$

A vector $X_u\in TV^{n+m}$, tangent in the point $u\in V^{n+m}$, is locally represented as $(x,y,X,\widetilde{X})=(x^i,y^a,X^i,X^a)$,where $(X^i)\in\mathbb{R}^n$ and $(X^a)\in\mathbb{R}^m$ are defined by the equality $X_u=X^i\partial_i+X^a\partial_a$ [$\partial_\alpha=(\partial_i,\partial_a)$ are usual partial derivatives on respective coordinates x^i and y^a]. For instance, $\pi^T\left(x,y,X,\widetilde{X}\right)=(x,X)$ and the submanifold vV^{n+m} contains elements of type $\left(x,y,0,\widetilde{X}\right)$ and the local fibers of the vertical subbundle are isomorphic to $\mathbb{R}^m$. Having $\pi^T(\partial_a)=0$, one comes out that ∂_a is a local basis of the vertical distribution $u\to v_uV^{n+m}$ on V^{n+m}, which is an integrable distribution.

[2] see Refs. [40, 14] for original results and constructions on vector and tangent bundles.

Definition 2.10. *A nonlinear connection (N–connection)* $\mathbf{N}$ *in a space* (V^{n+m}, π, V^n) *is defined by the splitting on the left of the exact sequence*

$$0 \to vV^{n+m} \to TV^{n+m}/vV^{n+m} \to 0, \tag{13}$$

i. e. a morphism of manifolds $N : TV^{n+m} \to vV^{n+m}$ *such that* $C \circ i$ *is the identity on* vV^{n+m}.

The kernel of the morphism $\mathbf{N}$ is a subbundle of $(TV^{n+m}, \tau_E, V^{n+m})$, it is called the horizontal subspace (being a subbundle for vector bundle constructions) and denoted by $(hV^{n+m}, \tau_H, V^{n+m})$. Every tangent bundle $(TV^{n+m},\ \tau_E,\ V^{n+m})$ provided with a N–connection structure is a Whitney sum of the vertical and horizontal subspaces (in brief, h- and v– subspaces), i. e.

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \tag{14}$$

It is proven that for every vector bundle (V^{n+m}, π, V^n) over a compact manifold V^n there exists a nonlinear connection [14] (the proof is similar if the bundle structure is modeled on a manifold).[3]

A N–connection $\mathbf{N}$ is defined locally by a set of coefficients $N_i^a(u^\alpha) = N_i^a(x^j, y^b)$ transforming as

$$N_{i'}^{a'} \frac{\partial x^{i'}}{\partial x^i} = M_a^{a'} N_i^a - \frac{\partial M_a^{a'}}{\partial x^i} y^a \tag{15}$$

under coordinate transforms on the space (V^{n+m}, μ, M) when $x^{i'} = x^{i'}(x^i)$ and $y^{a'} = M_a^{a'}(x) y^a$. The well known class of linear connections consists a particular parametization of its coefficients N_i^a to be linear on variables y^b,

$$N_i^a(x^j, y^b) = \Gamma_{bi}^a(x^j) y^b.$$

A N–connection structure can be associated to a prescribed ansatz of vielbein transforms

$$A_\alpha^{\ \underline{\alpha}}(u) = \mathbf{e}_\alpha^{\ \underline{\alpha}} = \begin{bmatrix} e_i^{\ \underline{i}}(u) & N_i^b(u) e_b^{\ \underline{a}}(u) \\ 0 & e_a^{\ \underline{a}}(u) \end{bmatrix}, \tag{16}$$

$$A^\beta_{\ \underline{\beta}}(u) = \mathbf{e}^\beta_{\ \underline{\beta}} = \begin{bmatrix} e^i_{\ \underline{i}}(u) & -N_k^b(u) e^k_{\ \underline{i}}(u) \\ 0 & e^a_{\ \underline{a}}(u) \end{bmatrix}, \tag{17}$$

in particular case $e_i^{\ \underline{i}} = \delta_i^{\underline{i}}$ and $e_a^{\ \underline{a}} = \delta_a^{\underline{a}}$ with $\delta_i^{\underline{i}}$ and $\delta_a^{\underline{a}}$ being the Kronecker symbols, defining a global splitting of $\mathbf{V}^{n+m}$ into "horizontal" and "vertical" subspaces with the N–vielbein structure

$$\mathbf{e}_\alpha = \mathbf{e}_\alpha^{\ \underline{\alpha}} \partial_{\underline{\alpha}} \text{ and } \vartheta^\beta = \mathbf{e}^\beta_{\ \underline{\beta}} du^{\underline{\beta}}.$$

In this work, we adopt the convention that for the spaces provided with N–connection structure the geometrical objects can be denoted by "boldfaced" symbols if it would be

[3] We note that the exact sequence (13) defines the N–connection in a global coordinate free form. In a similar form, the N–connection can be defined for covector bundles or, as particular cases for (co) tangent bundles. Generalizations for superspaces and noncommutative spaces are considered respectively in Refs. [23] and [36, 37].

necessary to dinstinguish such objects from similar ones for spaces without N–connection. The results from subsection 2.1 can be redefined in order to be compatible with the N–connection structure and rewritten in terms of "boldfaced" values.

A N–connection $\mathbf{N}$ in a space $\mathbf{V}^{n+m}$ is parametrized, with respect to a local coordinate base,

$$\partial_\alpha = (\partial_i, \partial_a) \equiv \frac{\partial}{\partial u^\alpha} = \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} \right), \tag{18}$$

and dual base (cobase),

$$d^\alpha = (d^i, d^a) \equiv du^\alpha = (dx^i, dy^a), \tag{19}$$

by its components $N_i^a(u) = N_i^a(x,y)$,

$$\mathbf{N} = N_i^a(u) d^i \otimes \partial_a.$$

It is characterized by the N–connection curvature $\mathbf{\Omega} = \{\Omega_{ij}^a\}$ as a Nijenhuis tensor field $N_v(X,Y)$ associated to $\mathbf{N}$,

$$\mathbf{\Omega} = N_v = [vX, vY] + v[X,Y] - v[vX,Y] - v[X,vY],$$

for $X,Y \in \mathcal{X}(V^{n+m})$ [41] and $[,]$ denoting commutators. In local form one has

$$\mathbf{\Omega} = \frac{1}{2} \Omega_{ij}^a d^i \wedge d^j \otimes \partial_a,$$

$$\Omega_{ij}^a = \delta_{[j} N_{i]}^a = \frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}. \tag{20}$$

The 'N–elongated' operators δ_j from (20) are defined from a certain vielbein configuration induced by the N–connection, the N–elongated partial derivatives (in brief, N–derivatives)

$$\mathbf{e}_\alpha \doteqdot \delta_\alpha = (\delta_i, \partial_a) \equiv \frac{\delta}{\delta u^\alpha} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a(u)\,\partial_a, \frac{\partial}{\partial y^a} \right) \tag{21}$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^\beta \doteqdot \delta^{\ \beta} = \left(d^i, \delta^a\right) \equiv \delta u^\alpha = \left(\delta x^i = dx^i, \delta y^a = dy^a + N_i^a(u)\,dx^i\right) \tag{22}$$

called also, respectively, the N–frame and N–coframe. [4]

The N–coframe (22) is anholonomic because there are satisfied the anholonomy relations (3),

$$[\delta_\alpha, \delta_\beta] = \delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\,\delta_\gamma \tag{23}$$

for which the anholonomy coefficients $\mathbf{w}_{\beta\gamma}^\alpha(u)$ are computed to have certain nontrivial values

$$\mathbf{w}^a{}_{ji} = -\mathbf{w}^a{}_{ij} = \Omega_{ij}^a,\ \ \mathbf{w}^b{}_{ia} = -\mathbf{w}^b{}_{ai} = \partial_a N_i^b. \tag{24}$$

[4]We shall use both type of denotations $\mathbf{e}_\alpha \doteqdot \delta_\alpha$ and $\vartheta^\beta \doteqdot \delta^{\ \alpha}$ in order to preserve a connection to denotations from Refs. [14, 23, 24, 34, 35, 25, 37]. The 'boldfaced' symbols $\mathbf{e}_\alpha$ and ϑ^β are written in order to emphasize that they define N–adapted vielbeins and the symbols δ_α and $\delta^{\ \beta}$ will be used for the N–elongated partial derivatives and, respectively, differentials.

We emphasize that the N–connection formalism is a natural one for investigating physical systems with mixed sets of holonomic–anholonomic variables. The imposed anholonomic constraints (anisotropies) are characterized by the coefficients of N–connection which defines a global splitting of the components of geometrical objects with respect to some 'horizontal' (holonomic) and 'vertical' (anisotropic) directions. In brief, we shall use respectively the terms h- and/or v–components, h- and/or v–indices, and h- and/or v–subspaces

A N–connection structure on $\mathbf{V}^{n+m}$ defines the algebra of tensorial distinguished (by N–connection structure) fields $dT\left(T\mathbf{V}^{n+m}\right)$ (d–fields, d–tensors, d–objects, if to follow the terminology from [14]) on $\mathbf{V}^{n+m}$ introduced as the tensor algebra $\mathcal{T}=\{\mathcal{T}_{qs}^{pr}\}$ of the distinguished tangent bundle $\mathcal{V}_{(d)}$, $p_d: \ h\mathbf{V}^{n+m}\oplus v\mathbf{V}^{n+m}\rightarrow\mathbf{V}^{n+m}$. An element $\mathbf{t}\in\mathcal{T}_{qs}^{pr}$, a d–tensor field of type $\left(\begin{array}{cc} p & r \\ q & s \end{array}\right)$, can be written in local form as

$$\mathbf{t}=t_{j_1\dots j_q b_1\dots b_r}^{i_1\dots i_p a_1\dots a_r}\left(u\right)\delta_{i_1}\otimes\dots\otimes\delta_{i_p}\otimes\partial_{a_1}\otimes\dots\otimes\partial_{a_r}\otimes d^{j_1}\otimes\dots\otimes d^{j_q}\otimes\delta^{b_1}\dots\otimes\delta^{b_r}.$$

There are used the denotations $\mathcal{X}\left(\mathcal{V}_{(d)}\right)$ (or $\mathcal{X}(\mathbf{V}^{n+m}),\wedge^p\left(\mathcal{V}_{(d)}\right)$ (or $\wedge^p\left(\mathbf{V}^{n+m}\right)$ and $\mathcal{F}\left(\mathcal{V}_{(d)}\right)$ (or $\mathcal{F}\ (\mathbf{V}^{n+m})$) for the module of d–vector fields on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$), the exterior algebra of p–forms on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$) and the set of real functions on $\mathcal{V}_{(d)}$ (or $\mathbf{V}^{n+m}$).

2.3 Distinguished linear connection and metric structures

The d–objects on $\mathcal{V}_{(d)}$ are introduced in a coordinate free form as geometric objects adapted to the N–connection structure. In coordinate form, we can characterize such objects (linear connections, metrics or any tensor field) by certain group and coordinate transforms adapted to the N-connection structure on $\mathbf{V}^{n+m}$, i. e. to the global space splitting (14) into h- and v–subspaces.

2.3.1 d–connections

We analyze the general properties of a class of linear connections being adapted to the N–connection structure (called d–connections).

Definition 2.11. *A d–connection $\mathbf{D}$ on $\mathcal{V}_{(d)}$ is defined as a linear connection D, see Definition 2.1, on $\mathcal{V}_{(d)}$ conserving under a parallelism the global decomposition of $T\mathbf{V}^{n+m}$ (14) into the horizontal subbundle, $h\mathbf{V}^{n+m}$, and vertical subbundle, $v\mathbf{V}^{n+m}$, of $\mathcal{V}_{(d)}$.*

A N-connection induces decompositions of d–tensor indices into sums of horizontal and vertical parts, for example, for every d–vector $\mathbf{X}\in\mathcal{X}\left(\mathcal{V}_{(d)}\right)$ and 1-form $\widetilde{\mathbf{X}}\in\Lambda^1\left(\mathcal{V}_{(d)}\right)$ we have respectively

$$X=hX+vX \text{ and } \quad \widetilde{X}=h\widetilde{X}+v\widetilde{X}.$$

For simplicity, we shall not use boldface symbols for d–vectors and d–forms if this will not result in ambiguities. In consequence, we can associate to every d–covariant derivation $\mathbf{D}_X=X\rfloor\mathbf{D}$ two new operators of h- and v–covariant derivations, $\mathbf{D}_X=D_X^{[h]}+D_X^{[v]}$, defined respectively

$$D_X^{[h]}Y=\mathbf{D}_{hX}Y \quad \text{ and } \quad D_X^{[v]}Y=\mathbf{D}_{vX}Y,$$

for which the following conditions hold:

$$
\begin{aligned}
\mathbf{D}_X Y &= D_X^{[h]} Y + D_X^{[v]} Y, \\
D_X^{[h]} f &= (hX) f \quad \text{ and } \quad D_X^{[v]} f = (vX) f,
\end{aligned} \tag{25}
$$

for any $X, Y \in \mathcal{X}(E), f \in \mathcal{F}(V^{n+m})$.

The N–adapted components $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ of a d-connection $\mathbf{D}_\alpha = (\delta_\alpha \rfloor \mathbf{D})$ are defined by the equations

$$
\mathbf{D}_\alpha \delta_\beta = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} \delta_\gamma,
$$

from which one immediately follows

$$
\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}(u) = (\mathbf{D}_\alpha \delta_\beta) \rfloor \delta^\gamma. \tag{26}
$$

The operations of h- and v-covariant derivations, $D_k^{[h]} = \{L^i_{jk}, L^a_{bk}\}$ and $D_c^{[v]} = \{C^i_{jk}, C^a_{bc}\}$ (see (25)) are introduced as corresponding h- and v–parametrizations of (26),

$$
L^i_{jk} = (\mathbf{D}_k \delta_j) \rfloor d^i, \quad L^a_{bk} = (\mathbf{D}_k \partial_b) \rfloor \delta^a \tag{27}
$$

$$
C^i_{jc} = (\mathbf{D}_c \delta_j) \rfloor d^i, \quad C^a_{bc} = (\mathbf{D}_c \partial_b) \rfloor \delta^a. \tag{28}
$$

A set of h–components (27) and v–components (28), distinguished in the form $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$, completely defines the local action of a d–connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$. For instance, having taken a d–tensor field of type $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $\mathbf{t} = t^{ia}_{jb} \delta_i \otimes \partial_a \otimes \partial^j \otimes \delta^b$, and a d–vector $\mathbf{X} = X^i \delta_i + X^a \partial_a$ we can write

$$
\mathbf{D}_X \mathbf{t} = D_X^{[h]} \mathbf{t} + D_X^{[v]} \mathbf{t} = \left(X^k t^{ia}_{jb|k} + X^c t^{ia}_{jb\perp c} \right) \delta_i \otimes \partial_a \otimes d^j \otimes \delta^b,
$$

where the h–covariant derivative is

$$
t^{ia}_{jb|k} = \frac{\delta t^{ia}_{jb}}{\delta x^k} + L^i_{hk} t^{ha}_{jb} + L^a_{ck} t^{ic}_{jb} - L^h_{jk} t^{ia}_{hb} - L^c_{bk} t^{ia}_{jc}
$$

and the v–covariant derivative is

$$
t^{ia}_{jb\perp c} = \frac{\partial t^{ia}_{jb}}{\partial y^c} + C^i_{hc} t^{ha}_{jb} + C^a_{dc} t^{id}_{jb} - C^h_{jc} t^{ia}_{hb} - C^d_{bc} t^{ia}_{jd}.
$$

For a scalar function $f \in \mathcal{F}(V^{n+m})$ we have

$$
D_k^{[h]} = \frac{\delta f}{\delta x^k} = \frac{\partial f}{\partial x^k} - N^a_k \frac{\partial f}{\partial y^a} \text{ and } D_c^{[v]} f = \frac{\partial f}{\partial y^c}.
$$

We note that these formulas are written in abstract index form and specify for d–connections the covariant derivation rule (1).

2.3.2 Metric structures and d–metrics

We introduce arbitrary metric structures on a space $\mathbf{V}^{n+m}$ and consider the possibility to adapt them to N–connection structures.

Definition 2.12. *A metric structure* $\mathbf{g}$ *on a space* $\mathbf{V}^{n+m}$ *is defined as a symmetric covariant tensor field of type* $(0,2)$, $g_{\alpha\beta}$, *being nondegenerate and of constant signature on* $\mathbf{V}^{n+m}$.

This Definition is completely similar to Definition 2.6 but in our case it is adapted to the N–connection structure. A N–connection $\mathbf{N} = \{N_{\underline{i}}^{\underline{b}}(u)\}$ and a metric structure

$$\mathbf{g} = g_{\underline{\alpha}\underline{\beta}} du^{\underline{\alpha}} \otimes du^{\underline{\beta}} \tag{29}$$

on $\mathbf{V}^{n+m}$ are mutually compatible if there are satisfied the conditions

$$\mathbf{g}\left(\delta_{\underline{i}}, \partial_{\underline{a}}\right) = 0, \text{ or equivalently, } g_{\underline{i}\underline{a}}(u) - N_{\underline{i}}^{\underline{b}}(u)\, h_{\underline{a}\underline{b}}(u) = 0, \tag{30}$$

where $h_{\underline{a}\underline{b}} \doteqdot \mathbf{g}\left(\partial_{\underline{a}}, \partial_{\underline{b}}\right)$ and $g_{\underline{i}\underline{a}} \doteqdot \mathbf{g}\left(\partial_{\underline{i}}, \partial_{\underline{a}}\right)$ resulting in

$$N_i^b(u) = h^{ab}(u)\, g_{ia}(u) \tag{31}$$

(the matrix h^{ab} is inverse to h_{ab}; for simplicity, we do not underly the indices in the last formula). In consequence, we obtain a h–v–decomposition of metric (in brief, d–metric)

$$\mathbf{g}(X,Y) = h\mathbf{g}(X,Y) + v\mathbf{g}(X,Y), \tag{32}$$

where the d-tensor $h\mathbf{g}(X,Y) = \mathbf{g}(hX, hY)$ is of type $\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and the d-tensor $v\mathbf{g}(X,Y) = \mathbf{h}(vX, vY)$ is of type $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$. With respect to a N–coframe (22), the d–metric (32) is written

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\, \delta^{\alpha} \otimes \delta^{\beta} = g_{ij}(u)\, d^i \otimes d^j + h_{ab}(u)\, \delta^a \otimes \delta^b, \tag{33}$$

where $g_{ij} \doteqdot \mathbf{g}\left(\delta_i, \delta_j\right)$. The d–metric (33) can be equivalently written in "off–diagonal" form if the basis of dual vectors consists from the coordinate differentials (19),

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b h_{ab} & N_j^e h_{ae} \\ N_i^e h_{be} & h_{ab} \end{bmatrix}. \tag{34}$$

It is easy to check that one holds the relations

$$\mathbf{g}_{\alpha\beta} = \mathbf{e}_{\alpha}^{\ \underline{\alpha}} \mathbf{e}_{\beta}^{\ \underline{\beta}} g_{\underline{\alpha}\underline{\beta}}$$

or, inversely,

$$g_{\underline{\alpha}\underline{\beta}} = \mathbf{e}^{\alpha}_{\ \underline{\alpha}} \mathbf{e}^{\beta}_{\ \underline{\beta}} \mathbf{g}_{\alpha\beta}$$

as it is stated by respective vielbein transforms (16) and (17).

Remark 2.1. *A metric, for instance, parametrized in the form (34) is generic off–diagonal if it can not be diagonalized by any coordinate transforms. If the anholonomy coefficients (24) vanish for a such parametrization, we can define certain coordinate transforms to diagonalize both the off–diagonal form (34) and the equivalent d–metric (33).*

Definition 2.13. *The nonmetricity d–field*

$$\mathcal{Q} = \mathbf{Q}_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta} = \mathbf{Q}_{\alpha\beta}\delta^{\ \alpha} \otimes \delta^{\beta}$$

on a space $\mathbf{V}^{n+m}$ provided with N–connection structure is defined by a d–tensor field with the coefficients

$$\mathbf{Q}_{\alpha\beta} \doteqdot -\mathbf{D}\mathbf{g}_{\alpha\beta} \tag{35}$$

where the covariant derivative $\mathbf{D}$ is for a d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, see (26) with the respective splitting $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^{i}_{jk}, L^{a}_{bk}, C^{i}_{jc}, C^{a}_{bc}\right)$, as to be adapted to the N–connection structure.

This definition is similar to that given for metric–affine spaces (see definition 2.8) and Refs. [4], but in our case the N–connection establishes some 'preferred' N–adapted local frames (21) and (22) splitting all geometric objects into irreducible h- and v–components. A linear connection D_X is compatible with a d–metric $\mathbf{g}$ if

$$D_X\mathbf{g} = 0, \tag{36}$$

$\forall X \in \mathcal{X}\left(V^{n+m}\right)$, i. e. if $Q_{\alpha\beta} \equiv 0$. In a space provided with N–connection structure, the metricity condition (36) may split into a set of compatibility conditions on h- and v–subspaces. We should consider separately which of the conditions

$$D^{[h]}(h\mathbf{g}) = 0, D^{[v]}(h\mathbf{g}) = 0, D^{[h]}(v\mathbf{g}) = 0, D^{[v]}(v\mathbf{g}) = 0 \tag{37}$$

are satisfied, or not, for a given d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$. For instance, if $D^{[v]}(h\mathbf{g}) = 0$ and $D^{[h]}(v\mathbf{g}) = 0$, but, in general, $D^{[h]}(h\mathbf{g}) \neq 0$ and $D^{[v]}(v\mathbf{g}) \neq 0$ we can consider a nonmetricity d–field (d–nonmetricity) $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ with irreducible h–v–components (with respect to the N–connection decompositions), $\mathbf{Q}_{\gamma\alpha\beta} = \left(Q_{ijk}, Q_{abc}\right)$.

By acting on forms with the covariant derivative D, in a metric–affine space, we can also define another very important geometric objects (the 'gravitational field potentials', see [4]):

$$\text{torsion } \ \mathcal{T}^{\alpha} \doteqdot D\vartheta^{\alpha} = d\vartheta^{\alpha} + \Gamma^{\gamma}{}_{\beta} \wedge \vartheta^{\beta}, \text{ see Definition 2.3} \tag{38}$$

and

$$\text{curvature } \ \mathcal{R}^{\alpha}{}_{\beta} \doteqdot D\Gamma^{\alpha}{}_{\beta} = d\Gamma^{\alpha}{}_{\beta} - \Gamma^{\gamma}{}_{\beta} \wedge \Gamma^{\alpha}{}_{\gamma}, \text{ see Definition 2.4.} \tag{39}$$

The Bianchi identities are

$$DQ_{\alpha\beta} \equiv \mathcal{R}_{\alpha\beta} + \mathcal{R}_{\beta\alpha}, \ D\mathcal{T}^{\alpha} \equiv \mathcal{R}_{\gamma}{}^{\alpha} \wedge \vartheta^{\gamma} \text{ and } D\mathcal{R}_{\gamma}{}^{\alpha} \equiv 0, \tag{40}$$

where we stress the fact that $Q_{\alpha\beta}, T^{\alpha}$ and $R_{\beta\alpha}$ are called also the strength fields of a metric–affine theory.

For spaces provided with N–connections, we write the corresponding formulas by using "boldfaced" symbols and change the usual differential d into N-adapted operator δ.

$$\mathbf{T}^{\alpha} \doteqdot \mathbf{D}\vartheta^{\alpha} = \delta\vartheta^{\alpha} + \mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \vartheta^{\beta} \quad (41)$$

and

$$\mathbf{R}^{\alpha}{}_{\beta} \doteqdot \mathbf{D}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \delta\mathbf{\Gamma}^{\alpha}{}_{\beta} - \mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \mathbf{\Gamma}^{\alpha}{}_{\gamma} \quad (42)$$

where the Bianchi identities written in 'boldfaced' symbols split into h- and v–irreducible decompositions induced by the N–connection. [5] We shall examine and compute the general form of torsion and curvature d–tensors in spaces provided with N–connection structure in section 2.4.

We note that the bulk of works on Finsler geometry and generalizations [15, 14, 20, 16, 17, 19, 13, 23, 24, 37] consider very general linear connection and metric fields being addapted to the N–connection structure. In another turn, the researches on metric–affine gravity [4, 38] concern generalizations to nonmetricity but not N–connections. In this work, we elaborate a unified moving frame geometric approach to both Finlser like and metric–affine geometries.

2.4 Torsions and curvatures of d–connections

We define and calculate the irreducible components of torsion and curvature in a space $\mathbf{V}^{n+m}$ provided with additional N–connection structure (these could be any metric–affine spaces [4], or their particular, like Riemann–Cartan [38], cases with vanishing nonmetricity and/or torsion, or any (co) vector / tangent bundles like in Finsler geometry and generalizations).

2.4.1 d–torsions and N–connections

We give a definition being equivalent to (41) but in d–operator form (the Definition 2.3 was for the spaces not possessing N–connection structure):

Definition 2.14. *The torsion* $\mathbf{T}$ *of a d–connection* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in space* $\mathbf{V}^{n+m}$ *is defined as an operator (d–tensor field) adapted to the N–connection structure*

$$\mathbf{T}(X,Y) = \mathbf{D}_X Y - \mathbf{D}_Y X - [X,Y]. \quad (43)$$

One holds the following h- and v–decompositions

$$\mathbf{T}(X,Y) = \mathbf{T}(hX,hY) + \mathbf{T}(hX,vY) + \mathbf{T}(vX,hY) + \mathbf{T}(vX,vY). \quad (44)$$

We consider the projections: $h\mathbf{T}(X,Y), v\mathbf{T}(hX,hY), h\mathbf{T}(hX,hY), ...$ and say that, for instance, $h\mathbf{T}(hX,hY)$ is the h(hh)-torsion of D , $vT(hX,hY)$ is the v(hh)-torsion of $\mathbf{D}$ and so on.

[5] see similar details in Ref. [14] for the case of vector/tangent bundles provided with mutually compatible N–connection, d–connection and d–metric structure.

The torsion (43) is locally determined by five d–tensor fields, d–torsions (irreducible N–adapted h–v–decompositions) defined as

$$\begin{aligned} T^i_{jk} &= h\mathbf{T}\left(\delta_k, \delta_j\right)\rfloor d^i, \quad T^a_{jk} = v\mathbf{T}\left(\delta_k, \delta_j\right)\rfloor \delta^a, \quad P^i_{jb} = h\mathbf{T}\left(\partial_b, \delta_j\right)\rfloor d^i, \\ P^a_{jb} &= v\mathbf{T}\left(\partial_b, \delta_j\right)\rfloor \delta^a, \quad S^a_{bc} = v\mathbf{T}\left(\partial_c, \partial_b\right)\rfloor \delta^a. \end{aligned}$$

Using the formulas (21), (22), and (20), we can calculate the h–v–components of torsion (44) for a d–connection, i. e. we can prove [6]

Theorem 2.2. *The torsion* $\mathbf{T}^{\alpha}_{.\beta\gamma} = (T^i_{.jk}, T^i_{ja}, T^a_{.ij}, T^a_{.bi}, T^a_{.bc})$ *of a d–connection* $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ *(26) has irreducible h- v–components (d–torsions)*

$$\begin{aligned} T^i_{.jk} &= -T^i_{kj} = L^i_{jk} - L^i_{kj}, \quad T^i_{ja} = -T^i_{aj} = C^i_{.ja}, \; T^a_{.ji} = -T^a_{.ij} = \frac{\delta N^a_i}{\delta x^j} - \frac{\delta N^a_j}{\delta x^i} = \Omega^a_{.ji}, \\ T^a_{.bi} &= -T^a_{.ib} = P^a_{.bi} = \frac{\partial N^a_i}{\partial y^b} - L^a_{.bj}, \; T^a_{.bc} = -T^a_{.cb} = S^a_{.bc} = C^a_{bc} - C^a_{cb}. \end{aligned} \tag{45}$$

We note that on (pseudo) Riemanian spacetimes the d–torsions can be induced by the N–connection coefficients and reflect an anholonomic frame structures. Such objects vanishes when we transfer our considerations with respect to holonomic bases for a trivial N–connection and zero "vertical" dimension.

2.4.2 d–curvatures and N–connections

In operator form, the curvature (42) is stated from the

Definition 2.15. *The curvature* $\mathbf{R}$ *of a d–connection* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in space* $\mathbf{V}^{n+m}$ *is defined as an operator (d–tensor field) adapted to the N–connection structure*

$$\mathbf{R}\left(X, Y\right) Z = \mathbf{D}_X \mathbf{D}_Y Z - \mathbf{D}_Y \mathbf{D}_X Z - \mathbf{D}_{[X,Y]} Z. \tag{46}$$

This Definition is similar to the Definition 2.4 being a generalization for the spaces provided with N–connection. One holds certain properties for the h- and v–decompositions of curvature:

$$v\mathbf{R}\left(X, Y\right) hZ = 0, \;\; h\mathbf{R}\left(X, Y\right) vZ{=}0, \;\; \mathbf{R}\left(X, Y\right) Z = h\mathbf{R}\left(X, Y\right) hZ{+}v\mathbf{R}\left(X, Y\right) vZ.$$

From (46) and the equation $\mathbf{R}\left(X, Y\right) = -\mathbf{R}\left(Y, X\right)$, we get that the curvature of a d-connection $\mathbf{D}$ in $\mathbf{V}^{n+m}$ is completely determined by the following six d–tensor fields (d–curvatures):

$$\begin{aligned} R^i{}_{hjk} &= d^i \rfloor \mathbf{R}\left(\delta_k, \delta_j\right) \delta_h, \;\; R^a{}_{bjk} = \delta^a \rfloor \mathbf{R}\left(\delta_k, \delta_j\right) \partial_b, \\ P^i{}_{jkc} &= d^i \rfloor \mathbf{R}\left(\partial_c, \partial_k\right) \delta_j, \;\; P^a{}_{bkc} = \delta^a \rfloor \mathbf{R}\left(\partial_c, \partial_k\right) \partial_b, \\ S^i{}_{jbc} &= d^i \rfloor \mathbf{R}\left(\partial_c, \partial_b\right) \delta_j, \;\; S^a{}_{bcd} = \delta^a \rfloor \mathbf{R}\left(\partial_d, \partial_c\right) \partial_b. \end{aligned} \tag{47}$$

By a direct computation, using (21), (22), (27), (28) and (47), we prove

[6] see also the original proof for vector bundles in [14]

Theorem 2.3. *The curvature* $\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = ({R^i}_{hjk}, {R^a}_{bjk}, {P^i}_{jka}, {P^c}_{bka}, {S^i}_{jbc}, {S^a}_{bcd})$ *of a d–connection* $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ *(26) has irreducible h- v–components (d–curvatures)*

$$\begin{aligned}
{R^i}_{hjk} &= \frac{\delta L^i_{.hj}}{\delta x^k} - \frac{\delta L^i_{.hk}}{\delta x^j} + L^m_{.hj}L^i_{mk} - L^m_{.hk}L^i_{mj} - C^i_{.ha}\Omega^a_{.jk}, \\
{R^a}_{bjk} &= \frac{\delta L^a_{.bj}}{\delta x^k} - \frac{\delta L^a_{.bk}}{\delta x^j} + L^c_{.bj}L^a_{.ck} - L^c_{.bk}L^a_{.cj} - C^a_{.bc}\ \Omega^c_{.jk}, \\
{P^i}_{jka} &= \frac{\partial L^i_{.jk}}{\partial y^k} - \left(\frac{\partial C^i_{.ja}}{\partial x^k} + L^i_{.lk}C^l_{.ja} - L^l_{.jk}C^i_{.la} - L^c_{.ak}C^i_{.jc}\right) + C^i_{.jb}P^b_{.ka}, \\
{P^c}_{bka} &= \frac{\partial L^c_{.bk}}{\partial y^a} - \left(\frac{\partial C^c_{.ba}}{\partial x^k} + L^c_{.dk}C^d_{.ba} - L^d_{.bk}C^c_{.da} - L^d_{.ak}C^c_{.bd}\right) + C^c_{.bd}P^d_{.ka}, \\
{S^i}_{jbc} &= \frac{\partial C^i_{.jb}}{\partial y^c} - \frac{\partial C^i_{.jc}}{\partial y^b} + C^h_{.jb}C^i_{.hc} - C^h_{.jc}C^i_{hb}, \\
{S^a}_{bcd} &= \frac{\partial C^a_{.bc}}{\partial y^d} - \frac{\partial C^a_{.bd}}{\partial y^c} + C^e_{.bc}C^a_{.ed} - C^e_{.bd}C^a_{.ec}.
\end{aligned} \tag{48}$$

The components of the Ricci d-tensor

$$\mathbf{R}_{\alpha\beta} = {\mathbf{R}^{\tau}}_{\alpha\beta\tau}$$

with respect to a locally adapted frame (21) has four irreducible h- v–components, $\mathbf{R}_{\alpha\beta} = \{R_{ij}, R_{ia}, R_{ai}, S_{ab}\}$, where

$$\begin{aligned}
R_{ij} &= {R^k}_{ijk}, \quad R_{ia} = -\ {}^2P_{ia} = -{P^k}_{ika}, \\
R_{ai} &= {}^1P_{ai} = {P^b}_{aib}, \quad S_{ab} = {S^c}_{abc}.
\end{aligned} \tag{49}$$

We point out that because, in general, ${}^1P_{ai} \neq {}^2P_{ia}$ the Ricci d–tensor is non symmetric.

Having defined a d–metric of type (33) in $\mathbf{V}^{n+m}$, we can introduce the scalar curvature of a d–connection $\mathbf{D}$,

$$\overleftarrow{\mathbf{R}} = \mathbf{g}^{\alpha\beta}\mathbf{R}_{\alpha\beta} = R + S, \tag{50}$$

where $R = g^{ij}R_{ij}$ and $S = h^{ab}S_{ab}$ and define the distinguished form of the Einstein tensor (the Einstein d–tensor), see Definition 2.7,

$$\mathbf{G}_{\alpha\beta} \doteqdot \mathbf{R}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\overleftarrow{\mathbf{R}}. \tag{51}$$

The Ricci and Bianchi identities (40) of d–connections are formulated in h- v- irreducible forms on vector bundle [14]. The same formulas hold for arbitrary metric compatible d–connections on $\mathbf{V}^{n+m}$ (for simplicity, we omit such details in this work).

3 Some Classes of Linear and Nonlinear Connections

The geometry of d–connections in a space $\mathbf{V}^{n+m}$ provided with N–connection structure is very reach (works [14] and [23] contain results on generalized Finsler spaces and superspaces). If a triple of fundamental geometric objects $\left(N^a_i(u), {\mathbf{\Gamma}^{\alpha}}_{\beta\gamma}(u), \mathbf{g}_{\alpha\beta}(u)\right)$ is fixed on

$\mathbf{V}^{n+m}$, in general, with respect to N–adapted frames, a multiconnection structure is defined (with different rules of covariant derivation). In this Section, we analyze a set of linear connections and associated covariant derivations being very important for investigating spacetimes provided with anholonomic frame structure and generic off–diagonal metrics.

3.1 The Levi–Civita connection and N–connections

The Levi–Civita connection $\bigtriangledown = \{\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}\}$ with coefficients

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = g\left(\mathbf{e}_{\alpha}, \bigtriangledown_{\gamma}\mathbf{e}_{\beta}\right) = \mathbf{g}_{\alpha\tau}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}, \tag{52}$$

is torsionless,

$$\mathbf{T}^{\alpha}_{\bigtriangledown} \doteqdot \bigtriangledown\vartheta^{\alpha} = d\vartheta^{\alpha} + \mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} \wedge \vartheta^{\beta} = 0,$$

and metric compatible, $\bigtriangledown\mathbf{g} = 0$, see see Definition 2.1. The formula (52) states that the operator $\bigtriangledown$ can be defined on spaces provided with N–connection structure (we use 'boldfaced' symbols) but this connection is not adapted to the N–connection splitting (14). It is defined as a linear connection but not as a d–connection, see Definition 2.11. The Levi–Civita connection is usually considered on (pseudo) Riemannian spaces but it can be also introduced, for instance, in (co) vector/tangent bundles both with respect to coordinate and anholonomic frames [14, 34, 35]. One holds a Theorem similar to the Theorem 2.1,

Theorem 3.1. *If a space $\mathbf{V}^{n+m}$ is provided with both N–connection $\mathbf{N}$ and d–metric $\mathbf{g}$ structures, there is a unique linear symmetric and torsionless connection $\bigtriangledown$, being metric compatible such that $\bigtriangledown_{\gamma}\mathbf{g}_{\alpha\beta} = 0$ for $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, see (33), with the coefficients*

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \mathbf{g}\left(\delta_{\alpha}, \bigtriangledown_{\gamma}\delta_{\beta}\right) = \mathbf{g}_{\alpha\tau}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma},$$

computed as

$$\mathbf{\Gamma}^{\bigtriangledown}_{\alpha\beta\gamma} = \frac{1}{2}\left[\delta_{\beta}\mathbf{g}_{\alpha\gamma} + \delta_{\gamma}\mathbf{g}_{\beta\alpha} - \delta_{\alpha}\mathbf{g}_{\gamma\beta} + \mathbf{g}_{\alpha\tau}\mathbf{w}^{\tau}_{\gamma\beta} + \mathbf{g}_{\beta\tau}\mathbf{w}^{\tau}_{\alpha\gamma} - \mathbf{g}_{\gamma\tau}\mathbf{w}^{\tau}_{\beta\alpha}\right] \tag{53}$$

with respect to N–frames $\mathbf{e}_{\beta} \doteqdot \delta_{\beta}$ (21) and N–coframes $\vartheta^{\alpha} \doteqdot \delta^{\alpha}$ (22).

The proof is that from Theorem 2.1, see also Refs. [45, 46], with $e_{\beta} \to \mathbf{e}_{\beta}$ and $\vartheta^{\beta} \to \vartheta^{\beta}$ substituted directly in formula (10).

With respect to coordinate frames ∂_{β} (18) and du^{α} (19), the metric (33) transforms equivalently into (29) with coefficients (34) and the coefficients of (53) transform into the usual Christoffel symbols (11). We emphasize that we shall use the coefficients just in the form (53) in order to compare the properties of different classes of connections given with respect to N–adapted frames. The coordinate form (11) is not "N–adapted", being less convenient for geometric constructions on spaces with anholonomic frames and associated N–connection structure.

We can introduce the 1-form formalism and express

$$\mathbf{\Gamma}^{\bigtriangledown}_{\gamma\alpha} = \mathbf{\Gamma}^{\bigtriangledown}_{\gamma\alpha\beta}\vartheta^{\beta}$$

where

$$\mathbf{\Gamma}^{\bigtriangledown}{}_{\gamma\alpha} = \frac{1}{2}\left[\mathbf{e}_{\gamma}\rfloor\ \delta\vartheta_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \delta\vartheta_{\gamma} - (\mathbf{e}_{\gamma}\rfloor\ \mathbf{e}_{\alpha}\rfloor\ \delta\vartheta_{\beta}) \wedge \vartheta^{\beta}\right], \tag{54}$$

contains h- v-components, $\mathbf{\Gamma}^{\gamma}_{\bigtriangledown\alpha\beta} = \left(L^{i}_{\bigtriangledown jk}, L^{a}_{\bigtriangledown bk}, C^{i}_{\bigtriangledown jc}, C^{a}_{\bigtriangledown bc}\right)$, defined similarly to (27) and (28) but using the operator $\bigtriangledown$,

$$L^{i}_{\bigtriangledown jk} = (\bigtriangledown_{k}\delta_{j})\rfloor d^{i}, \quad L^{a}_{\bigtriangledown bk} = (\bigtriangledown_{k}\partial_{b})\rfloor\delta^{a},\ C^{i}_{\bigtriangledown jc} = (\bigtriangledown_{c}\delta_{j})\rfloor d^{i}, \quad C^{a}_{\bigtriangledown bc} = (\bigtriangledown_{c}\partial_{b})\rfloor\delta^{a}.$$

In explicit form, the components $L^{i}_{\bigtriangledown jk}, L^{a}_{\bigtriangledown bk}, C^{i}_{\bigtriangledown jc}$ and $C^{a}_{\bigtriangledown bc}$ are defined by formula (54) if we consider N–frame $\mathbf{e}_{\gamma} = (\delta_{i} = \partial_{i} - N^{a}_{i}\partial_{a}, \partial_{a})$ and N–coframe $\vartheta^{\beta} = (dx^{i}, \delta y^{a} = dy^{a} + N^{a}_{i}dx^{i})$ and a d–metric $\mathbf{g} = (g_{ij,} h_{ab})$. In these formulas, we write $\delta\vartheta_{\alpha}$ instead of absolute differentials $d\vartheta_{\alpha}$ from Refs. [4, 38] because the N–connection is considered. The coefficients (54) transforms into the usual Levi–Civita (or Christoffel) ones for arbitrary anholonomic frames e_{γ} and ϑ^{β} and for a metric

$$g = g_{\alpha\beta}\vartheta^{\alpha} \otimes \vartheta^{\beta}$$

if $\mathbf{e}_{\gamma} \rightarrow e_{\gamma}$, $\vartheta^{\beta} \rightarrow \vartheta^{\beta}$ and $\delta\vartheta_{\beta} \rightarrow d\vartheta_{\beta}$.

Finally, we note that if the N–connection structure is not trivial, we can define arbitrary vielbein transforms starting from $\mathbf{e}_{\gamma}$ and ϑ^{β}, i. e. $e^{[N]}_{\alpha} = A_{\alpha}{}^{\alpha'}(u)\mathbf{e}_{\alpha'}$ and $\vartheta^{\beta}_{[N]} = A^{\beta}{}_{\beta'}(u)\vartheta^{\beta'}$ (we put the label $[N]$ in order to emphasize that such object were defined by vielbein transforms starting from certain N–adapted frames). This way we develop a general anholonomic frame formalism adapted to the prescribed N–connection structure. If we consider geometric objects with respect to coordinate frames $\mathbf{e}_{\alpha'} \rightarrow \partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}}$ and coframes $\vartheta^{\beta'} \rightarrow du^{\underline{\beta}}$, the N–connection strucuture is 'hidden' in the off–diagonal metric coefficients (34) and performed geometric constructions, in general, are not N–adapted.

3.2 The canonical d–connection and the Levi–Civita connection

The Levi–Civita connection $\bigtriangledown$ is constructed only from the metric coefficients, being torsionless and satisfying the metricity conditions $\bigtriangledown_{\alpha}g_{\beta\gamma} = 0$. Because the Levi–Civita connection is not adapted to the N–connection structure, we can not state its coefficients in an irreducible form for the h– and v–subspaces. We need a type of d–connection which would be similar to the Levi–Civita connection but satisfy certain metricity conditions adapted to the N–connection.

Proposition 3.1. *There are metric d–connections* $\mathbf{D} = \left(D^{[h]}, D^{[v]}\right)$ *in a space* $\mathbf{V}^{n+m}$, *see (25), satisfying the metricity conditions if and only if*

$$D^{[h]}_{k} g_{ij} = 0,\ D^{[v]}_{a} g_{ij} = 0,\ D^{[h]}_{k} h_{ab} = 0,\ D^{[h]}_{a} h_{ab} = 0. \tag{55}$$

The general proof of existence of such metric d–connections on vector (super) bundles is given in Ref. [14]. Here we note that the equations (55) on $\mathbf{V}^{n+m}$ are just the conditions (37). In our case the existence may be proved by constructing an explicit example:

Definition 3.1. *The canonical d–connection* $\widehat{\mathbf{D}} = \left(\widehat{D}^{[h]}, \widehat{D}^{[v]}\right)$, *equivalently* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, *is defined by the h– v–irreducible components* $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$,

$$
\begin{aligned}
\widehat{L}^{i}_{jk} &= \frac{1}{2}g^{ir}\left(\frac{\delta g_{jk}}{\delta x^{k}} + \frac{\delta g_{kr}}{\delta x^{j}} - \frac{\delta g_{jk}}{\delta x^{r}}\right), \qquad (56)\\
\widehat{L}^{a}_{bk} &= \frac{\partial N^{a}_{k}}{\partial y^{b}} + \frac{1}{2}h^{ac}\left(\frac{\delta h_{bc}}{\delta x^{k}} - \frac{\partial N^{d}_{k}}{\partial y^{b}}h_{dc} - \frac{\partial N^{d}_{k}}{\partial y^{c}}h_{db}\right),\\
\widehat{C}^{i}_{jc} &= \frac{1}{2}g^{ik}\frac{\partial g_{jk}}{\partial y^{c}},\\
\widehat{C}^{a}_{bc} &= \frac{1}{2}h^{ad}\left(\frac{\partial h_{bd}}{\partial y^{c}} + \frac{\partial h_{cd}}{\partial y^{b}} - \frac{\partial h_{bc}}{\partial y^{d}}\right).
\end{aligned}
$$

satisfying the torsionless conditions for the h–subspace and v–subspace, respectively, $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0.$

By straightforward calculations with (56) we can verify that the conditions (55) are satisfied and that the d–torsions are subjected to the conditions $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0$ (see section 2.4)). We emphasize that the canonical d–torsion posses nonvanishing torsion components,

$$
\widehat{T}^{a}_{.ji} = -\widehat{T}^{a}_{.ij} = \frac{\delta N^{a}_{i}}{\delta x^{j}} - \frac{\delta N^{a}_{j}}{\delta x^{i}} = \Omega^{a}_{.ji},\ \widehat{T}^{i}_{ja} = -\widehat{T}^{i}_{aj} = \widehat{C}^{i}_{.ja},\ \widehat{T}^{a}_{.bi} = -\widehat{T}^{a}_{.ib} = \widehat{P}^{a}_{.bi} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - \widehat{L}^{a}_{.bj}
$$

induced by $\widehat{L}^{a}_{bk}$, $\widehat{C}^{i}_{jc}$ and N–connection coefficients N^{a}_{i} and their partial derivatives $\partial N^{a}_{i}/\partial y^{b}$ (as is to be computed by introducing (56) in formulas (45)). This is an anholonmic frame effect.

Proposition 3.2. *The components of the Levi–Civita connection* $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}$ *and the irreducible components of the canonical d–connection* $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ *are related by formulas*

$$
\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk} - \frac{\partial N^{a}_{k}}{\partial y^{b}}, \widehat{C}^{i}_{jc} + \frac{1}{2}g^{ik}\Omega^{a}_{jk}h_{ca}, \widehat{C}^{a}_{bc}\right), \qquad (57)
$$

where Ω^{a}_{jk} *is the N–connection curvature (20).*

The proof follows from an explicit decomposition of N–adapted frame (21) and N–adapted coframe (22) in (53) (equivalently, in (54)) and regroupation of the components as to distinguish the h- and v– irreducible values (56) for $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$.

We conclude from (57) that, in a trivial case, the Levi–Civita and the canonical d–connection are given by the same h– v– components $\left(\widehat{L}^{i}_{jk}, \widehat{L}^{a}_{bk}, \widehat{C}^{i}_{jc}, \widehat{C}^{a}_{bc}\right)$ if $\Omega^{a}_{jk} = 0,$ and $\partial N^{a}_{k}/\partial y^{b} = 0$. This results in zero anholonomy coefficients (24) when the anholonomic N–basis is reduced to a holonomic one. It should be also noted that even in this case some components of the anholonomically induced by d–connection torsion $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$ could be nonzero (see formulas (95) just for $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$). For instance, one holds the

Corollary 3.1. *The d–tensor components*

$$\widehat{T}^{a}_{.bi} = -\widehat{T}^{a}_{.ib} = \widehat{P}^{a}_{.bi} = \frac{\partial N^a_i}{\partial y^b} - \widehat{L}^{a}_{.bj} \tag{58}$$

for a canonical d–connection (56) can be nonzero even $\partial N^a_k/\partial y^b = 0$ and $\Omega^a_{jk} = 0$ and a trivial equality of the components of the canonical d–connection and of the Levi–Civita connection, $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ holds just with respect to coordinate frames.

This quite surprising fact follows from the anholonomic character of the N–connection structure. If a N–connection is defined, there are imposed specific types of constraints on the frame structure. This is important for definition of d–connections (being adapted to the N–connection structure) but not for the Levi–Civita connection which is not a d–connection. Even such linear connections have the same components with respect to a N–adapted (co) frame, they are very different geometrical objects because they are subjected to different rules of transformation with respect to frame and coordinate transforms. The d–connections' transforms are adapted to those for the N–connection (15) but the Levi–Civita connection is subjected to general rules of linear connection transforms (2).[7]

Proposition 3.3. *A canonical d–connection $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ defined by a N–connection N^a_i and d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ has zero d–torsions (95) if an only if there are satisfied the conditions $\Omega^a_{jk} = 0, \widehat{C}^i_{jc} = 0$ and $\widehat{L}^a_{.bj} = \partial N^a_i/\partial y^b$, i. e. $\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk} = \partial N^a_i/\partial y^b, 0, \widehat{C}^a_{bc}\right)$ which is equivalent to*

$$g^{ik}\frac{\partial g_{jk}}{\partial y^c} = 0, \tag{59}$$

$$\frac{\delta h_{bc}}{\delta x^k} - \frac{\partial N^d_k}{\partial y^b} h_{dc} - \frac{\partial N^d_k}{\partial y^c} h_{db} = 0, \tag{60}$$

$$\frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b} = 0. \tag{61}$$

The Levi–Civita connection defined by the same N–connection and d–metric structure with respect to N–adapted (co) frames has the components ${}^{[0]}\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma} = \left(\widehat{L}^i_{jk}, 0, 0, \widehat{C}^a_{bc}\right)$.

Proof: The relations (59)–(61) follows from the condition of vanishing of d–torsion coefficients (95) when the coefficients of the canonical d–connection and the Levi–Civita connection are computed respectively following formulas (56) and (57)

[7]The Corollary 3.1 is important for constructing various classes of exact solutions with generic off–diagonal metrics in Einstein gravity, its higher dimension and/or different gauge, Einstein--Cartan and metric–affine generalizations. Certain type of ansatz were proven to result in completely integrable gravitational field equations just for the canonical d–connection (but not for the Levi–Civita one), see details in Refs. [34, 35, 25, 37]. The induced d–torsion (58) is contained in the Ricci d–tensor $R_{ai} = {}^1P_{ai} = P^{.b}_{a.ib}$, see (49), i. e. in the Einstein d–tensor constructed for the canonical d–connection. If a class of solutions were obtained for a d–connection, we can select those subclasses which satisfy the condition $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma} = \widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}$ with respect to a frame of reference. In this case the nontrivial d–torsion $\widehat{T}^a_{.bi}$ (58) can be treated as an object constructed from some "pieces" of a generic off–diagonal metric and related to certain components of the N–adapted anholonomic frames.

We note a specific separation of variables in the equations (59)–(61). For instance, the equation (59) is satisfied by any $g_{ij} = g_{ij}\left(x^k\right)$. We can search a subclass of N–connections with $N_j^a = \delta_j N^a$, i. e. of 1–forms on the h–subspace, $\widetilde{N}^a = \delta_j N^a dx^i$ which are closed on this subspace,

$$\delta \widetilde{N}^a = \frac{1}{2}\left(\frac{\partial N_i^a}{\partial x^j} - \frac{\partial N_j^a}{\partial x^i} + N_i^b \frac{\partial N_j^a}{\partial y^b} - N_j^b \frac{\partial N_i^a}{\partial y^b}\right) dx^i \wedge dx^j = 0,$$

satisfying the (61). Having defined such N_i^a and computing the values $\partial_c N_i^a$, we may try to solve (60) rewritten as a system of first order partial differential equations

$$\frac{\partial h_{bc}}{\partial x^k} = N_k^e \frac{\partial h_{bc}}{\partial y^e} + \partial_b N_k^d\ h_{dc} + \partial_c N_k^d h_{db}$$

with known coefficients.■

We can also associate the nontrivial values of $\widehat{\mathbf{T}}_{\beta\gamma}^{\tau}$ (in particular cases, of $\widehat{T}_{.bi}^{a}$) to be related to any algebraic equations in the Einstein–Cartan theory or dynamical equations for torsion like in string or supergravity models. But in this case we shall prescribe a specific class of anholonomically constrained dynamics for the N–adapted frames.

Finally, we note that if a (pseudo) Riemannian space is provided with a generic off–diagonal metric structure (see Remark 2.1) we can consider alternatively to the Levi–Civita connection an infinite number of metric d–connections, details in the section 3.5.1. Such d–connections have nontrivial d–torsions $\mathbf{T}_{\beta\gamma}^{\tau}$ induced by anholonomic frames and constructed from off–diagonal metric terms and h- and v–components of d–metrics.

3.3 The set of metric d–connections

Let us define the set of all possible metric d–connections, satisfying the conditions (55) and being constructed only form g_{ij}, h_{ab} and N_i^a and their partial derivatives. Such d–connections satisfy certain conditions for d–torsions that $T^i{}_{jk} = T^a{}_{bc} = 0$ and can be generated by two procedures of deformation of the connection

$$\begin{array}{rcl} \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} & \rightarrow & {}^{[K]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} + {}^{[K]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} \text{ (Kawaguchi's metrization [39])}, \\ \text{or} & \rightarrow & {}^{[M]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + {}^{[M]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} \text{ (Miron's connections [14])}. \end{array}$$

Theorem 3.2. *Every deformation d–tensor (equivalently, distorsion, or deflection)*

$$\begin{aligned} {}^{[K]}\mathbf{Z}^{\gamma}{}_{\alpha\beta} = \{ & {}^{[K]}Z^i{}_{jk} = \frac{1}{2} g^{im} D_j^{[h]} g_{mk},\ {}^{[K]}Z^a{}_{bk} = \frac{1}{2} h^{ac} D_k^{[h]} h_{cb}, \\ & {}^{[K]}Z^i{}_{ja} = \frac{1}{2} g^{im} D_a^{[v]} g_{mj},\ {}^{[K]}Z^a{}_{bc} = \frac{1}{2} h^{ad} D_c^{[v]} h_{db} \} \end{aligned}$$

transforms a d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a\right)$ *(26) into a metric d–connection*

$${}^{[K]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(L_{jk}^i + {}^{[K]}Z^i{}_{jk}, L_{bk}^a + {}^{[K]}Z^a{}_{bk}, C_{jc}^i + {}^{[K]}Z^i{}_{ja}, C_{bc}^a + {}^{[K]}Z^a{}_{bc}\right).$$

The proof consists from a straightforward verification which demonstrate that the conditions (55) are satisfied on $\mathbf{V}^{n+m}$ for ${}^{[K]}\mathbf{D} = \{{}^{[\mathbf{K}]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta}\}$ and $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$. We note that the Kawaguchi's metrization procedure contains additional covariant derivations of the d–metric coefficients, defined by arbitrary d–connection, not only N–adapted derivatives of the d–metric and N–connection coefficients as in the case of the canonical d–connection.

Theorem 3.3. *For a fixed d–metric structure (33),* $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$, *on a space* $\mathbf{V}^{n+m}$, *the set of metric d–connections* ${}^{[M]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + {}^{[M]}\mathbf{Z}^{\gamma}{}_{\alpha\beta}$ *is defined by the deformation d–tensor*

$$
\begin{aligned}
{}^{[M]}\mathbf{Z}^{\gamma}_{\alpha\beta} \;\; = \;\; \{ & \, {}^{[M]}Z^{i}{}_{jk} = {}^{[-]}O^{li}_{km} Y^{m}_{lj}, \; {}^{[M]}Z^{a}{}_{bk} = {}^{[-]}O^{ea}_{bd} Y^{m}_{ej}, \\
& {}^{[M]}Z^{i}{}_{ja} = {}^{[+]}O^{mi}_{jk} Y^{k}_{mc}, \; {}^{[M]}Z^{a}{}_{bc} = {}^{[+]}O^{ea}_{bd} Y^{d}_{ec} \}
\end{aligned}
$$

where the so–called Obata operators are defined

$$
{}^{[\pm]}O^{li}_{km} = \frac{1}{2}\left(\delta^{l}_{k}\delta^{i}_{m} \pm g_{km}g^{li}\right) \;\; \textit{and} \;\; {}^{[\pm]}O^{ea}_{bd} = \frac{1}{2}\left(\delta^{e}_{b}\delta^{a}_{d} \pm h_{bd}h^{ea}\right)
$$

and Y^{m}_{lj}, Y^{m}_{ej}, Y^{k}_{mc}, Y^{d}_{ec} *are arbitrary d–tensor fields.*

The proof consists from a direct verification of the fact that the conditions (55) are satisfied on $\mathbf{V}^{n+m}$ for ${}^{[M]}\mathbf{D} = \{{}^{[\mathbf{M}]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$. We note that the relation (57) between the Levi–Civita and the canonical d–connection is a particular case of ${}^{[M]}\mathbf{Z}^{\gamma}_{\alpha\beta}$, when Y^{m}_{lj}, Y^{m}_{ej} and Y^{d}_{ec} are zero, but Y^{k}_{mc} is taken to have ${}^{[+]}O^{mi}_{jk} Y^{k}_{mc} = \frac{1}{2} g^{ik} \Omega^{a}_{jk} h_{ca}$.

There is a very important consequence of the Theorems 3.2 and 3.3: For a generic off–diagonal metric structure (34) we can derive a N–connection structure N^{a}_{i} with a d–metric $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ (33). So, we may consider an infinite number of d–connections $\{\mathbf{D}\}$, all constructed from the coefficients of the off–diagonal metrics, satisfying the metricity conditions $\mathbf{D}_{\gamma}\mathbf{g}_{\alpha\beta} = 0$ and having partial vanishing torsions, $T^{i}_{jk} = T^{a}_{bc} = 0$. The covariant calculi associated to the set $\{\mathbf{D}\}$ are adapted to the N–connection splitting and alternative to the covariant calculus defined by the Levi–Civita connection $\bigtriangledown$, which is not adapted to the N–connection.

3.4 Nonmetricity in Finsler Geometry

Usually, the N–connection, d–connection and d–metric in generalized Finsler spaces satisfy certain metric compatibility conditions [14, 15, 16, 17]. Nevertheless, there were considered some classes of d–connections (for instance, related to the Berwald d–connection) with nontrivial components of the nonmetricity d–tensor. Let us consider some such examples modeled on metric–affine spaces.

3.4.1 The Berwald d–connection

A d–connection of Berwald type (see, for instance, Ref. [14] on such configurations in Finsler and Lagrange geometry), ${}^{[B]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[B]}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, is defined by h- and v–irreducible components

$$
{}^{[B]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(\widehat{L}^{i}{}_{jk}, \frac{\partial N^{a}_{k}}{\partial y^{b}}, 0, \widehat{C}^{a}{}_{bc}\right), \tag{62}
$$

with $\widehat{L}^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc}$ taken as in (56), satisfying only partial metricity compatibility conditions for a d–metric (33), $\mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab})$ on space $\mathbf{V}^{n+m}$

$$^{[B]}D_k^{[h]} g_{ij} = 0 \text{ and } \ ^{[B]}D_c^{[v]} h_{ab} = 0.$$

This is an example of d–connections which may possess nontrivial nonmetricity components, $^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = \left(^{[B]}Q_{cij},\ ^{[B]}Q_{iab}\right)$ with

$$^{[B]}Q_{cij} = \ ^{[B]}D_c^{[v]} g_{ij} \text{ and } \ ^{[B]}Q_{iab} = \ ^{[B]}D_i^{[h]} h_{ab}. \tag{63}$$

So, the Berwald d–connection defines a metric–affine space $\mathbf{V}^{n+m}$ with N–connection structure.

If $\widehat{L}^{i}{}_{jk} = 0$ and $\widehat{C}^{a}{}_{bc} = 0$, we obtain a Berwald type connection

$$^{[N]}\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(0, \frac{\partial N_k^a}{\partial y^b}, 0, 0\right)$$

induced by the N–connection structures. It defines a vertical covariant derivation $^{[N]}D_c^{[v]}$ acting in the v–subspace of $\mathbf{V}^{n+m}$, with the coefficients being partial derivatives on v–coordinates y^a of the N–connection coefficients N_i^a [41].

We can generalize the Berwald connection (62) to contain any fixed values of d–torsions $T^i_{.jk}$ and $T^a_{.bc}$ from the h- v–decomposition (95). We can check by a straightforward calculations that the d–connection

$$^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^{i}{}_{jk} + \tau^{i}{}_{jk}, \frac{\partial N_k^a}{\partial y^b}, 0, \widehat{C}^{a}{}_{bc} + \tau^{a}{}_{bc}\right) \tag{64}$$

with

$$\begin{aligned} \tau^{i}{}_{jk} &= \frac{1}{2} g^{il} \left(g_{kh} T^h_{.lj} + g_{jh} T^h_{.lk} - g_{lh} T^h{}_{jk}\right) \\ \tau^{a}{}_{bc} &= \frac{1}{2} h^{ad} \left(h_{bf} T^f{}_{dc} + h_{cf} T^f{}_{db} - h_{df} T^f{}_{bc}\right) \end{aligned} \tag{65}$$

results in $^{[B\tau]}\mathbf{T}^i_{jk} = T^i_{.jk}$ and $^{[B\tau]}\mathbf{T}^a_{bc} = T^a_{.bc}$. The d–connection (64) has certain nonvanishing irreducible nonmetricity components $^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left(^{[B\tau]}Q_{cij},\ ^{[B\tau]}Q_{iab}\right)$.

In general, by using the Kawaguchi metrization procedure (see Theorem 3.2) we can also construct metric d–connections with prescribed values of d–torsions $T^i_{.jk}$ and $T^a_{.bc}$, or to express, for instance, the Levi–Civita connection via coefficients of an arbitrary metric d–connection (see details, for vector bundles, in [14]).

Similarly to formulas (75), (76) and (77), we can express a general affine Berwald d–connection $^{[B\tau]}\mathbf{D}$, i. e. $^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \ ^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$, via its deformations from the Levi–Civita connection $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$,

$$^{[B\tau]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta} + \ ^{[B\tau]}\mathbf{Z}^{\alpha}{}_{\beta}, \tag{66}$$

$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ being expressed as (54) (equivalently, defined by (53)) and

$$\begin{aligned} ^{[B\tau]}\mathbf{Z}_{\alpha\beta} &= \mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ ^{[B\tau]}\mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\ &\quad + \left(\mathbf{e}_{\alpha}\rfloor\ ^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ ^{[B\tau]}\mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ ^{[B\tau]}\mathbf{Q}_{\alpha\beta}. \end{aligned} \tag{67}$$

defined with prescribed d–torsions ${}^{[B\tau]}\mathbf{T}^{i}_{jk} = T^{i}_{.jk}$ and ${}^{[B\tau]}\mathbf{T}^{a}_{bc} = T^{a}_{.bc}$. This Berwald d–connection can define a particular subclass of metric–affine connections being adapted to the N–connection structure and with prescribed values of d–torsions.

3.4.2 The canonical/ Berwald metric–affine d–connections

If the deformations of d–metrics in formulas (76) and (66) are considered not with respect to the Levi–Civita connection $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ but with respect to the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ with h- v–irreducible coefficients (56), we can construct a set of canonical metric–affine d–connections. Such metric–affine d–connections $\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ are defined via deformations

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} = \widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + \widehat{\mathbf{Z}}^{\alpha}{}_{\beta}, \tag{68}$$

$\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta}$ being the canonical d–connection (26) and

$$\begin{aligned}
\widehat{\mathbf{Z}}_{\alpha\beta} &= \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\
&\quad + \left(\mathbf{e}_{\alpha}\rfloor\ {}^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ {}^{[B\tau]}\mathbf{Q}_{\alpha\beta}
\end{aligned} \tag{69}$$

where $\mathbf{T}_{\alpha}$ and $\mathbf{Q}_{\alpha\beta}$ are arbitrary torsion and nonmetricity structures.

A metric–affine d–connection $\mathbf{\Gamma}^{\gamma}{}_{\alpha}$ can be also considered as a deformation from the Berwald connection ${}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$

$$\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} + {}^{[B\tau]}\ \widehat{\mathbf{Z}}^{\alpha}{}_{\beta}, \tag{70}$$

${}^{[B\tau]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ being the Berwald d–connection (64) and

$$\begin{aligned}
{}^{[B\tau]}\ \widehat{\mathbf{Z}}^{\alpha}{}_{\beta} &= \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} \\
&\quad + \left(\mathbf{e}_{\alpha}\rfloor\ {}^{[B\tau]}\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ {}^{[B\tau]}\mathbf{Q}_{\alpha\beta}
\end{aligned} \tag{71}$$

The h- and v–splitting of formulas can be computed by introducing N–frames $\mathbf{e}_{\gamma} = (\delta_i = \partial_i - N^a_i\partial_a, \partial_a)$ and N–coframes $\vartheta^{\beta} = (dx^i, \delta y^a = dy^a + N^a_i dx^i)$ and d–metric $\mathbf{g} = (g_{ij}, h_{ab})$ into (54), (66) and (67) for the general Berwald d–connections. In a similar form we can compute splitting by introducing the N–frames and d–metric into (26), (68) and (69) for the metric affine canonic d–connections and, respectively, into (64), (70) and (71) for the metric–affine Berwald d–connections. For the corresponding classes of d–connections, we can compute the torsion and curvature tensors by introducing respective connections (54), (76), (56), (62), (64), (66), (68) and (70) into the general formulas for torsion (41) and curvature (42) on spaces provided with N–connection structure.

3.5 N–connections in metric–affine spaces

In order to elaborate a unified MAG and generalized Finsler spaces scheme, it is necessary to explain how the N–connection emerge in a metric–affine space and/or in more particular cases of Riemann–Cartan and (pseudo) Riemann geometry.

3.5.1 Riemann geometry as a Riemann–Cartan geometry with N–connection

It is well known the interpretation of the Riemann–Cartan geometry as a generalization of the Riemannian geometry by distorsions (of the Levi–Civita connection) generated by the torsion tensors [38]. Usually, the Riemann–Cartan geometry is described by certain geometric relations between the torsion tensor, curvature tensor, metric and the Levi–Civita connection on effective Riemann spaces. We can establish new relations between the Riemann and Riemann–Cartan geometry if generic off–diagonal metrics and anholonomic frames of reference are introduced into consideration. Roughly speaking, a generic off–diagonal metric induces alternatively to the well known Riemann spaces a certain class of Riemann–Cartan geometries, with torsions completely defined by off–diagonal metric terms and related anholonomic frame structures.

Theorem 3.4. *Any (pseudo) Riemannian spacetime provided with a generic off–diagonal metric, defining the torsionless and metric Levi–Civita connection, can be equivalently modeled as a Riemann–Cartan spacetime provided with a canonical d–connection adapted to N–connection structure.*

Proof:

Let us consider how the data for a (pseudo) Riemannian generic off–diagonal metric $g_{\alpha\beta}$ parametrized in the form (34) can generate a Riemann–Cartan geometry. It is supposed that with respect to any convenient anholonomic coframes (22) the metric is transformed into a diagonalized form of type (33), which gives the possibility to define N_i^a and $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ and to compute the aholonomy coefficients $\mathbf{w}^{\gamma}{}_{\alpha\beta}$ (24) and the components of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left(\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc}\right)$ (56). This connection has nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{.\beta\gamma}$, see the Theorem 2.2 and Corollary 3.1. In general, such d–torsions are not zero being induced by the values N_i^a and their partial derivatives, contained in the former off–diagonal components of the metric (34). So, the former Riemannian geometry, with respect to anholonomic frames with associated N–connection structure, is equivalently rewritten in terms of a Riemann–Cartan geometry with nontrivial torsion structure.

We can provide an inverse construction when a diagonal d–metric (33) is given with respect to an anholonomic coframe (22) defined from nontrivial values of N–connection coefficients, N_i^a. The related Riemann–Cartan geometry is defined by the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ possessing nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{.\beta\gamma}$. The data for this geometry with N–connection and torsion can be directly transformed [even with respect to the same N–adapted (co) frames] into the data of related (pseudo) Riemannian geometry by using the relation (57) between the components of $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ and of the Levi–Civita connection $\mathbf{\Gamma}^{\tau}_{\bigtriangledown\beta\gamma}$.■

Remark 3.1.
a) Any generic off–diagonal (pseudo) Riemannian metric $g_{\alpha\beta}[N_i^a] \rightarrow \mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ *induces an infinite number of associated Riemann–Cartan geometries defined by sets of d–connections* $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ *which can be constructed according the Kawaguchi's and, respectively, Miron's Theorems 3.2 and 3.3.*
b) For any metric d–connection $\mathbf{D} = \{\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\}$ *induced by a generic off–diagonal metric (34), we can define alternatively to the standard (induced by the Levi–Civita connection) the Ricci d–tensor (49),* $\mathbf{R}_{\alpha\beta}$*, and the Einstein d–tensor (51),* $\mathbf{G}_{\alpha\beta}$*.*

We emphasize that all Riemann–Cartan geometries induced by metric d–connections $\mathbf{D}$ are characterized not only by nontrivial induced torsions $\mathbf{T}^{\alpha}_{.\beta\gamma}$ but also by corresponding nonsymmetric Ricci d–tensor, $\mathbf{R}_{\alpha\beta}$, and Einstein d–tensor, $\mathbf{G}_{\alpha\beta}$, for which $\mathbf{D}_{\gamma}\mathbf{G}_{\alpha\beta} \neq 0$. This is not a surprising fact, because we transferred the geometrical and physical objects on anholonomic spaces, when the conservation laws should be redefined as to include the anholonomically imposed constraints.

Finally, we conclude that for any generic off–diagonal (pseudo) Riemannian metric we have two alternatives: 1) to choose the approach defined by the Levi–Civita connection $\bigtriangledown$, with vanishing torsion and usually defined conservation laws $\bigtriangledown_{\gamma}\mathbf{G}^{[\bigtriangledown]}_{\alpha\beta} = 0$, or 2) to diagonalize the metric effectively, by respective anholonomic transforms, and transfer the geometric and physical objects into effective Riemann–Cartan geometries defined by corresponding N–connection and d–connection structures. All types of such geometric constructions are equivalent. Nevertheless, one could be defined certain priorities for some physical models like "simplicity" of field equations and definition of conservation laws and/or the possibility to construct exact solutions. We note also that a variant with induced torsions is more appropriate for including in the scheme various type of generalized Finsler structures and/or models of (super) string gravity containing nontrivial torsion fields.

3.5.2 Metric–affine geometry and N–connections

A general affine (linear) connection $D = \bigtriangledown + Z = \{\Gamma^{\gamma}_{\beta\alpha} = \Gamma^{\gamma}_{\bigtriangledown\beta\alpha} + Z^{\gamma}_{\beta\alpha}\}$

$$\Gamma^{\gamma}{}_{\alpha} = \Gamma^{\gamma}_{\alpha\beta}\vartheta^{\beta}, \tag{72}$$

can always be decomposed into the Riemannian $\Gamma^{\alpha}_{\bigtriangledown\ \beta}$ and post–Riemannian $Z^{\alpha}_{\ \beta}$ parts (see Refs. [4] and, for irreducible decompositions to the effective Einstein theory, see Ref. [26]),

$$\Gamma^{\alpha}_{\ \beta} = \Gamma^{\alpha}_{\bigtriangledown\ \beta} + Z^{\alpha}_{\ \beta} \tag{73}$$

where the distorsion 1-form $Z^{\alpha}_{\ \beta}$ is expressed in terms of torsion and nonmetricity,

$$Z_{\alpha\beta} = e_{\beta}\rfloor T_{\alpha} - e_{\alpha}\rfloor T_{\beta} + \frac{1}{2}\left(e_{\alpha}\rfloor e_{\beta}\rfloor T_{\gamma}\right)\vartheta^{\gamma} + \left(e_{\alpha}\rfloor Q_{\beta\gamma}\right)\vartheta^{\gamma} - \left(e_{\beta}\rfloor Q_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}Q_{\alpha\beta}, \tag{74}$$

T_{α} is defined as (38) and $Q_{\alpha\beta} \doteqdot -Dg_{\alpha\beta}$. [8] For $Q_{\beta\gamma} = 0$, we obtain from (74) just the distorsion for the Riemannian–Cartan geometry [38].

By substituting arbitrary (co) frames, metrics and linear connections into N–adapted ones (i. e. performing changes

$$e_{\alpha} \rightarrow \mathbf{e}_{\alpha}, \vartheta^{\beta} \rightarrow \vartheta^{\beta}, g_{\alpha\beta} \rightarrow \mathbf{g}_{\alpha\beta} = (g_{ij}, h_{ab}), \Gamma^{\gamma}{}_{\alpha} \rightarrow \mathbf{\Gamma}^{\gamma}{}_{\alpha}$$

with $\mathbf{Q}_{\alpha\beta} = \mathbf{Q}_{\gamma\alpha\beta}\vartheta^{\gamma}$ and $\mathbf{T}^{\alpha}$ as in (41)) into respective formulas (72), (73) and (74), we can define an affine connection $\mathbf{D} = \bigtriangledown + \mathbf{Z} = \{\mathbf{\Gamma}^{\gamma}{}_{\beta\alpha}\}$ with respect to N–adapted (co) frames,

$$\mathbf{\Gamma}^{\gamma}{}_{\alpha} = \mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}, \tag{75}$$

[8] We note that our $\Gamma^{\gamma}{}_{\alpha}$ and $Z^{\alpha}_{\ \beta}$ are respectively the $\Gamma^{\gamma}{}_{\alpha}$ and $N_{\alpha\beta}$ from Ref. [26]; in our works we use the symbol N for N–connections.

with

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta} + \mathbf{Z}^{\alpha}_{\ \ \beta}, \tag{76}$$

$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ being expressed as (54) (equivalently, defined by (53)) and $\mathbf{Z}^{\alpha}_{\ \ \beta}$ expressed as

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\mathbf{Q}_{\alpha\beta}. \tag{77}$$

The h– and v–components of $\mathbf{\Gamma}^{\alpha}_{\ \beta}$ from (76) consists from the components of $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta}$ (considered for (54)) and of $\mathbf{Z}_{\alpha\beta}$ with $\mathbf{Z}^{\alpha}_{\ \ \gamma\beta} = \left(Z^{i}_{jk}, Z^{a}_{bk}, Z^{i}_{jc}, Z^{a}_{bc}\right)$ .The values

$$\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\gamma\beta} + \mathbf{Z}^{\alpha}_{\ \ \gamma\beta} = \left(L^{i}_{\bigtriangledown jk} + Z^{i}_{jk}, L^{a}_{\bigtriangledown bk} + Z^{a}_{bk}, C^{i}_{\bigtriangledown jc} + Z^{i}_{jc}, C^{a}_{\bigtriangledown bc} + Z^{a}_{bc}\right)$$

are defined correspondingly

$$\begin{aligned} L^{i}_{\bigtriangledown jk} + Z^{i}_{\ jk} &= [(\bigtriangledown_{k} + Z_{k})\delta_{j}]\rfloor d^{i}, \quad L^{a}_{\bigtriangledown bk} + Z^{a}_{\ bk} = [(\bigtriangledown_{k} + Z_{k})\partial_{b}]\rfloor\delta^{a}, \\ C^{i}_{\bigtriangledown jc} + Z^{i}_{\ jc} &= [(\bigtriangledown_{c} + Z_{c})\,\delta_{j}]\rfloor d^{i}, \quad C^{a}_{\bigtriangledown bc} + Z^{a}_{\ bc} = [(\bigtriangledown_{c} + Z_{c})\,\partial_{b}]\rfloor\delta^{a}. \end{aligned}$$

and related to (77) via h- and v–splitting of N–frames $\mathbf{e}_{\gamma} = (\delta_{i} = \partial_{i} - N^{a}_{i}\partial_{a}, \partial_{a})$ and N–coframes $\vartheta^{\beta} = (dx^{i}, \delta y^{a} = dy^{a} + N^{a}_{i}dx^{i})$ and d–metric $\mathbf{g} = (g_{ij,}h_{ab})$.

We note that for $\mathbf{Q}_{\alpha\beta} = 0$, the distorsion 1–form $\mathbf{Z}_{\alpha\beta}$ defines a Riemann–Cartan geometry adapted to the N–connection structure.

Let us briefly outline the procedure of definition of N–connections in a metric–affine space V^{n+m} with arbitrary metric and connection structures $\left(g^{[od]} = \{g_{\alpha\beta}\}, \underline{\Gamma}^{\gamma}_{\ \ \beta\alpha}\right)$ and show how the geometric objects may be adapted to the N–connection structure.

Proposition 3.4. *Every metric–affine space provided with a generic off–diagonal metric structure admits nontrivial N–connections.*

Proof: We give an explicit example how to introduce the N–connection structure. We write the metric with respect to a local coordinate basis,

$$g^{[od]} = g_{\underline{\alpha}\underline{\beta}}du^{\underline{\alpha}} \otimes du^{\underline{\beta}},$$

where the matrix $g_{\underline{\alpha}\underline{\beta}}$ contains a non–degenerated $(m \times m)$ submatrix h_{ab}, for instance like in ansatz (34). Having fixed the block h_{ab}, labeled by running of indices $a, b, ... = n+1, n+2, ..., n+m$, we can define the $(n \times n)$ bloc $g_{\underline{i}\underline{j}}$ with indices $\underline{i}, \underline{j}, ... = 1, 2, ...n$. The next step is to find any nontrivial N^{a}_{i} (the set of coefficients has being defined, we may omit underlying) and find $N^{\underline{e}}_{\underline{j}}$ from the $(n \times m)$ block relations $g_{\underline{j}\underline{a}} = N^{\underline{e}}_{\underline{j}}h_{\underline{a}\underline{e}}$. This is always possible if $g_{\underline{\alpha}\underline{\beta}}$ is generic off–diagonal. The next step is to compute $g_{ij} = g_{\underline{i}\underline{j}} - N^{\underline{a}}_{\underline{i}}N^{\underline{e}}_{\underline{j}}h_{\underline{a}\underline{e}}$ which gives the possibility to transform equivalently

$$g^{[od]} \rightarrow \mathbf{g} = g_{ij}\vartheta^{i} \otimes \vartheta^{j} + h_{ab}\vartheta^{a} \otimes \vartheta^{b}$$

where

$$\vartheta^{i} \doteqdot dx^{i},\ \vartheta^{a} \doteqdot \delta y^{a} = dy^{a} + N^{a}_{i}\,(u)\,dx^{i}$$

are just the N–elongated differentials (22) if the local coordinates associated to the block h_{ab} are denoted by y^{a} and the rest ones by x^{i}. We impose a global splitting of the metric–affine spacetime by stating that all geometric objects are subjected to anholonomic frame

transforms with vielbein coefficients of type (16) and (17) defined by $\mathbf{N} = \{N_i^a\}$. This way, we define on the metric–affine space a vector/covector bundle structure if the coordinates y^a are treated as certain local vector/ covector components.■

We note, that having defined the values $\vartheta^\alpha = \left(\vartheta^i, \vartheta^b\right)$ and their duals $\mathbf{e}_\alpha = (\mathbf{e}_i, \mathbf{e}_a)$, we can compute the linear connection coefficients with respect to N–adapted (co) frames, $\Gamma^\gamma{}_{\beta\alpha} \to \widetilde{\mathbf{\Gamma}}^\gamma{}_{\beta\alpha}$. However, $\widetilde{\mathbf{\Gamma}}^\gamma{}_{\beta\alpha}$, in general, is not a d–connection, i. e. it is not adapted to the global splitting $T\mathbf{V}^{n+m} = h\mathbf{V}^{n+m} \oplus v\mathbf{V}^{n+m}$ defined by N–connection, see Definition 2.11. If the metric and linear connection are not subjected to any field equations, we are free to consider distorsion tensors in order to be able to apply the Theorems 3.2 and/or 3.3 with the aim to transform $\widetilde{\mathbf{\Gamma}}^\gamma{}_{\beta\alpha}$ into a metric d–connection, or even into a Riemann–Cartan d–connection. Here, we also note that a metric–affine space, in general, admits different classes of N–connections with various nontrivial global splittings $n' + m' = n + m$, where $n' \neq n$.

We can state from the very beginning that a metric–affine space $\mathbf{V}^{n+m}$ is provided with d–metric (33) and d–connection strucuture (26) adapted to a class of prescribed vielbein transforms (16) and (17) and N–elongated frames (21) and (22). All constructions can be redefined with respect to coordinate frames (18) and (19) with off–diagonal metric parametrization (34) and then subjected to another frame and coordinate transforms hiding the existing N–connection structure and distinguished character of geometric objects. Such 'distinguished' metric–affine spaces are characterized by corresponding N–connection geometries and admit geometric constructions with distinguished objects. They form a particular subclass of metric–affine spaces admitting transformations of the general linear connection $\Gamma^\gamma{}_{\beta\alpha}$ into certain classes of d–connections $\mathbf{\Gamma}^\gamma{}_{\beta\alpha}$.

Definition 3.2. *A distinguished metric–affine space* $\mathbf{V}^{n+m}$ *is a usual metric–affine space additionally enabled with a N–connection structure* $\mathbf{N} = \{N_i^a\}$ *inducing splitting into respective irreducible horizontal and vertical subspaces of dimensions* n *and* m. *This space is provided with independent d–metric (33) and affine d–connection (26) structures adapted to the N–connection.*

The metric–affine spacetimes with stated N–connection structure are also characterized by nontrivial anholonomy relations of type (23) with anholonomy coefficients (24). This is a very specific type of noncommutative symmetry generated by N–adapted (co) frames defining different anholonomic noncommutative differential calculi (for details with respect to the Einstein and gauge gravity see Ref. [47]).

We construct and analyze explicit examples of metric–affine spacetimes with associated N–connection (noncommutative) symmetry in Refs. [33]. A surprizing fact is that various types of d–metric ansatz (33) with associated N–elongated frame (21) and coframe (22) (or equivalently, respective off–diagonal ansatz (34)) can be defined as exact solutions in Einstein gravity of different dimensions and in metric–affine, or Einstein–Cartan gravity and gauge model realizations. Such solutions model also generalized Finsler structures.

4 Generalized Finsler–Affine Spaces

The aim of this section is to demonstrate that any well known type of locally anistoropic or locally isotropic spaces can be defined as certain particular cases of distinguished metric–

affine spaces. We use the general term of "generalized Finsler–affine spaces" for all type of geometries modeled in MAG as generalizations of the Riemann–Cartan–Finsler geometry, in general, containing nonmetricity fields. A complete classification of such spaces is given by Tables 1–11 in the Appendix.

4.1 Spaces with vanishing N–connection curvature

Three examples of such spaces are given by the well known (pseudo) Riemann, Riemann–Cartan or Kaluza–Klein manifolds of dimension $(n+m)$ provided with a generic off–diagonal metric structure $\underline{g}_{\alpha\beta}$ of type (34), of corresponding signature, which can be reduced equivalently to the block $(n\times n)\oplus(m\times m)$ form (33) via vielbein transforms (16). Their N–connection structures may be restricted by the condition $\Omega^a_{ij}=0$, see (20).

4.1.1 Anholonomic (pseudo) Riemannian spaces

The (pseudo) Riemannian manifolds, $\mathbf{V}_R^{n+m}$, provided with a generic off–diagonal metric and anholonomic frame structrue effectively diagonalizing such a metric is an anholonomic (pseudo) Riemannian space. The space admits associated N–connection structures with coefficients induced by generic off–diagonal terms in the metric (34). If the N–connection curvature vanishes, the Levi–Civita connection is closely defined by the same coefficients as the canonical d–connection (linear connections computed with respect to the N–adapted (co) frames), see Proposition 3.2 and related discussions in section 3. Following the Theorem 3.4, any (pseudo) Riemannian space enabled with generic off–diagonal connection structure can be equivalently modeled as a effective Riemann–Cartan geometry with induced N–connection and d–torsions.

There were constructed a number of exact 'off–diagonal' solutions of the Einstein equations [34, 35, 25], for instance, in five dimensional gravity (with various type restrictions to lower dimensions) with nontrivial N–connection structure with ansatz for metric of type

$$\begin{aligned}\mathbf{g} &= \omega\left(x^i,y^4\right)[g_1(dx^1)^2+g_2\left(x^2,x^3\right)(dx^2)^2+g_3\left(x^2,x^3\right)(dx^3)^2 \\ &\quad +h_4\left(x^2,x^3,y^4\right)\left(\delta y^4\right)^2+h_5\left(x^2,x^3,y^4\right)\left(\delta y^5\right)^2],\end{aligned}\tag{78}$$

for $g_1=const$, where

$$\delta y^a = dy^a + N^a_k\left(x^i,y^4\right)dy^a$$

with indices $i,j,k...=1,2,3$ and $a=4,5.$ The coefficients $N^a_i\left(x^i,y^4\right)$ were searched as a metric ansatz of type (34) transforming equivalently into a certain diagonalized block (33) would parametrize generic off–diagonal exact solutions. Such effective N–connections are contained into a corresponding anholonomic moving or static configuration of tetrads/ pentads (vierbeins/funfbeins) defining a conventional splitting of coordinates into n holonomic and m anholonomic ones, where $n+m=4,5.$ The ansatz (78) results in exact solutions of vacuum and nonvacuum Einstein equations which can be integrated in general form. Perhaps, all known at present time exact solutions in 3-5 dimensional gravity can be included as particular cases in (78) and generalized to anholonomic configurations with running constants and gravitational and matter polarizations (in general, anisotropic on variable y^4) of the metric and frame coefficients.

The vector/ tangent bundle configurations and/or torsion structures can be effectively modeled on such (pseudo) Riemannian spaces by prescribing a corresponding class of anholonomic frames. Such configurations are very different from those, for instance, defined by Killing symmetries and the induced torsion vanishes after frame transforms to coordinate bases. For a corresponding parametrizations of $N_i^a(u)$ and $g_{\alpha\beta}$, we can model Finsler like structures even in (pseudo) Riemannian spacetimes or in gauge gravity [25, 36, 37].

The anholonomic Riemannian spaces $\mathbf{V}_R^{n+m}$ consist a subclass of distinguished metric–affine spaces $\mathbf{V}^{n+m}$ provided with N–connection structure, characterized by the condition that nonmetricity d–filed $\mathbf{Q}_{\alpha\beta\gamma} = 0$ and that a certain type of induced torsions $\mathbf{T}_{\beta\gamma}^{\alpha}$ vanish for the Levi–Civita connection. We can take a generic off–diagonal metric (34), transform it into a d–metric (33) and compute the h- and v-components of the canonical d–connection (26) and put them into the formulas for d–torsions (95) and d–curvatures (48). The vacuum solutions are defined by d–metrics and N–connections satisfying the condition $\mathbf{R}_{\alpha\beta} = 0$, see the h–, v–components (49).

In order to transform certain geometric constructions defined by the canonical d–connection into similar ones for the Levi–Civita connection, we have to constrain the N–connection structure as to have vanishing N–curvature, $\Omega_{ij}^a = 0$, or to see the conditions when the deformation of Levi–Civita connection to any d–connection result in non–deformations of the Einstein equation. We obtain a (pseudo) Riemannian vacuum spacetime with anholonomially induced d–torsion components $\widehat{T}_{ja}^i = -\widehat{T}_{aj}^i = \widehat{C}_{.ja}^i$ and $\widehat{T}_{.bi}^a = -\widehat{T}_{.ib}^a = \partial N_i^a/\partial y^b - \widehat{L}_{.bj}^a$. This torsion can be related algebraicaly to a spin source like in the usual Riemann–Cartan gravity if we want to give an algebraic motivation to the N–connection splitting. We emphasize that the N–connection and d–metric coefficients can be chosen in order to model on $\mathbf{V}_R^{n+m}$ a special subclass of Finsler/ Lagrange structures (see discussion in section 4.2.3).

4.1.2 Kaluza–Klein spacetimes

Such higher dimension generalizations of the Einstein gravity are characterized by a metric ansatz

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{ij}(x^\kappa) + A_i^a(x^\kappa)A_j^b(x^\kappa)h_{ab}(x^\kappa, y^a) & A_j^e(x^\kappa)h_{ae}(x^\kappa, y^a) \\ A_i^e(x^\kappa)h_{be}(x^\kappa, y^a) & h_{ab}(x^\kappa, y^a) \end{array} \right] \quad (79)$$

(a particular case of the metric (34)) with certain compactifications on extra dimension coordinates y^a. The values $A_i^a(x^\kappa)$ are considered to define gauge fields after compactifications (the electromatgnetic potential in the original extension to five dimensions by Kaluza and Klein, or some non–Abelian gauge fields for higher dimension generalizations). Perhaps, the ansatz (79) was originally introduced in Refs. [48] (see [49] as a review of non–supersymmetry models and [50] for supersymmetric theories).

The coefficients $A_i^a(x^\kappa)$ from (79) are certain particular parametrizations of the N–connection coefficients $N_i^a(x^\kappa, y^a)$ in (34). This suggests a physical interpretation for the N–connection as a specific nonlinear gauge field depending both on spacetime and extra dimension coordinates (in general, noncompactified). In the usual Kaluza–Klein (super) theories, there were not considered anholonomic transforms to block d–metrics (33) containing dependencies on variables y^a.

In some more general approaches, with additional anholonomic structures on lower dimensional spacetime, there were constructed a set of exact vacuum five dimensional solutions by reducing ansatz (79) and their generalizations of form (34) to d–metric ansatz of type (78), see Refs. [34, 35, 36, 37, 25, 47]. Such vacuum and nonvacuum solutions describe anisotropically polarized Taub–NUT spaces, wormhole/ flux tube confiugurations, moving four dimensional black holes in bulk five dimensional spacetimes, anisotropically deformed cosmological spacetimes and various type of locally anisotropic spinor–soliton–dialton interactions in generalized Kaluza–Klein and string/ brane gravity.

4.1.3 Teleparallel spaces

Teleparallel theories are usually defined by two geometrical constraints [9] (here, we introduce them for d–connections and nonvanishing N–connection structure),

$$\mathbf{R}^{\alpha}_{\ \beta} = \delta \mathbf{\Gamma}^{\alpha}_{\ \beta} + \mathbf{\Gamma}^{\alpha}_{\ \gamma} \wedge \mathbf{\Gamma}^{\gamma}_{\ \beta} = 0 \tag{80}$$

and

$$\mathbf{Q}_{\alpha\beta} = -\mathbf{D}\mathbf{g}_{\alpha\beta} = -\delta \mathbf{g}_{\alpha\beta} + \mathbf{\Gamma}^{\gamma}_{\ \beta}\mathbf{g}_{\alpha\gamma} + \mathbf{\Gamma}^{\gamma}_{\ \alpha}\mathbf{g}_{\beta\gamma} = 0. \tag{81}$$

The conditions (80) and (81) establish a distant paralellism in such spaces because the result of a parallel transport of a vector does not depend on the path (the angles and lengths being also preserved under parallel transports). It is always possible to find such anholonomic transforms $e_\alpha = A_\alpha^{\ \underline{\beta}} e_{\underline{\beta}}$ and $e_{\underline{\alpha}} = A_{\underline{\alpha}}^{\ \beta} e_\beta$, where $A_{\underline{\alpha}}^{\ \beta}$ is inverse to $A_\alpha^{\ \underline{\beta}}$ when

$$\mathbf{\Gamma}^{\alpha}_{\ \beta} \rightarrow \mathbf{\Gamma}^{\underline{\alpha}}_{\ \underline{\beta}} = A_{\underline{\beta}}^{\ \beta} \mathbf{\Gamma}^{\alpha}_{\ \beta} A^{\underline{\alpha}}_{\ \alpha} + A^{\underline{\alpha}}_{\ \gamma} \delta A_{\underline{\beta}}^{\ \gamma} = 0$$

and the transformed local metrics becomes the standard Minkowski,

$$g_{\underline{\alpha\beta}} = diag\,(-1, +1, ..., +1)$$

(it can be fixed any signature). If the (co) frame is considered as the only dynamical variable, it is called that the space (and choice of gauge) are of Weitzenbock type. A coframe of type (22)

$$\vartheta^{\beta} \doteqdot \left(\delta x^i = dx^i, \delta y^a = dy^a + N_i^a\,(u)\,dx^i\right)$$

is defined by N–connection coefficients. If we impose the condition of vanishing the N–connection curvature, $\Omega^a_{ij} = 0$, see (20), the N–connection defines a specific anholonomic dynamics because of nontrivial anholonomic relations (23) with nonzero components (24).

By embedding teleparallel configurations into metric–affine spaces provided with N–connection structure we state a distinguished class of (co) frame fields adapted to this structure and open possibilities to include such spaces into Finsler–affine ones, see section 4.2.4. For vielbein fields $\mathbf{e}_\alpha^{\ \underline{\alpha}}$ and their inverses $\mathbf{e}^{\alpha}_{\ \underline{\alpha}}$ related to the d–metric (33),

$$\mathbf{g}_{\alpha\beta} = \mathbf{e}_\alpha^{\ \underline{\alpha}} \mathbf{e}_\beta^{\ \underline{\beta}} g_{\underline{\alpha\beta}}$$

we define the Weitzenbock d–connection

$${}^{[W]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma} = \mathbf{e}^{\alpha}_{\ \underline{\alpha}} \delta_\gamma \mathbf{e}_\beta^{\ \underline{\alpha}}, \tag{82}$$

where δ_γ is the N–elongated partial derivative (21). It transforms in the usual Weitzenbock connection for trivial N–connections. The torsion of ${}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ is defined

$$
{}^{[W]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = {}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} - {}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\gamma\beta}. \tag{83}
$$

It posses h– and v–irreducible components constructed from the components of a d–metric and N–adapted frames. We can express

$$
{}^{[W]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}
$$

where $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$ is the Levi–Civita connection (53) and the contorsion tensor is

$$
\mathbf{Z}_{\alpha\beta} = \mathbf{e}_{\beta}\rfloor\ {}^{[W]}\mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ {}^{[W]}\mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ {}^{[W]}\mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\mathbf{Q}_{\alpha\beta}.
$$

In formulation of teleparallel alternatives to the general relativity it is considered that $\mathbf{Q}_{\alpha\beta} = 0$.

4.2 Finsler and Finsler–Riemann–Cartan spaces

The first approaches to Finsler spaces [15, 16] were developed by generalizing the usual Riemannian metric interval

$$
ds = \sqrt{g_{ij}\left(x\right)dx^{i}dx^{j}}
$$

on a manifold M of dimension n into a nonlinear one

$$
ds = F\left(x^{i}, dx^{j}\right) \tag{84}
$$

defined by the Finsler metric F (fundamental function) on $\widetilde{TM} = TM\backslash\{0\}$ (it should be noted an ambiguity in terminology used in monographs on Finsler geometry and on gravity theories with respect to such terms as Minkowski space, metric function and so on). It is also considered a quadratic form on $\mathbb{R}^2$ with coefficients

$$
g_{ij}^{[F]} \rightarrow h_{ab} = \frac{1}{2}\frac{\partial^{2}F^{2}}{\partial y^{i}\partial y^{j}} \tag{85}
$$

defining a positive definite matrix. The local coordinates are denoted $u^{\alpha} = (x^{i}, y^{a} \rightarrow y^{i})$. There are satisfied the conditions: 1) The Finsler metric on a real manifold M is a function $F : TM \rightarrow \mathbb{R}$ which on $\widetilde{TM} = TM\backslash\{0\}$ is of class C^{∞} and F is only continuous on the image of the null cross–sections in the tangent bundle to M. 2) $F\left(x, \chi y\right) = \chi F\left(x, y\right)$ for every $\mathbb{R}_{+}^{*}$. 3) The restriction of F to $\widetilde{TM}$ is a positive function. 4) $rank\left[g_{ij}^{[F]}(x, y)\right] = n$.

There were elaborated a number of models of locally anisotropic spacetime geometry with broken local Lorentz invariance (see, for instance, those based on Finsler geometries [17, 19]). In result, in the Ref. [51], it was ambigously concluded that Finsler gravity models are very restricted by experimental data. Recently, the subject concerning Lorentz symmetry violations was revived for instance in brane gravity [52] (see a detailed analysis and references on such theoretical and experimental researches in [53]). In this case,

the Finsler like geometries broking the local four dimensional Lorentz invariance can be considered as a possible alternative direction for investigating physical models both with local anisotropy and violation of local spacetime symmetries. But it should be noted here that violations of postulates of general relativity is not a generic property of the so–called "Finsler gravity". A subclass of Finsler geometries and their generalizations could be induced by anholonomic frames even in general relativity theory and Riemannian–Cartan or gauge gravity [25, 36, 37, 24]. The idea is that instead of geometric constructions based on straightforward applications of derivatives of (85), following from a nonlinear interval (84), we should consider d–metrics (33) with the coefficients from Finsler geometry (85) or their extended variants. In this case, certain type Finsler configurations can be defined even as exact 'off–diagonal' solutions in vacuum Einstein gravity or in string gravity.

4.2.1 Finsler geometry and its almost Kahlerian model

We outline a modern approach to Finsler geometry [14] based on the geometry of nonlinear connections in tangent bundles.

A real (commutative) Finsler space $\mathbf{F}^n = (M, F(x,y))$ can be modeled on a tangent bundle TM enabled with a Finsler metric $F(x^i, y^j)$ and a quadratic form $g_{ij}^{[F]}$ (85) satisfying the mentioned conditions and defining the Christoffel symbols (not those from the usual Riemannian geometry)

$$c_{jk}^{\iota}(x,y) = \frac{1}{2} g^{ih} \left(\partial_j g_{hk}^{[F]} + \partial_k g_{jh}^{[F]} - \partial_h g_{jk}^{[F]} \right),$$

where $\partial_j = \partial/\partial x^j$, and the Cartan nonlinear connection

$$^{[F]}\mathbf{N}_j^i(x,y) = \frac{1}{4} \frac{\partial}{\partial y^j} \left[c_{lk}^{\iota}(x,y) y^l y^k \right], \tag{86}$$

where we do not distinguish the v- and h- indices taking on TM the same values.

In Finsler geometry, there were investigated different classes of remarkable Finsler linear connections introduced by Cartan, Berwald, Matsumoto and other geometers (see details in Refs. [15, 17, 16]). Here we note that we can introduce $g_{ij}^{[F]} = g_{ab}$ and $^{[F]}\mathbf{N}_j^i(x,y)$ in (34) and transfer our considerations to a $(n \times n) \oplus (n \times n)$ blocks of type (33) for a metric–affine space V^{n+n}.

A usual Finsler space $\mathbf{F}^n = (M, F(x,y))$ is completely defined by its fundamental tensor $g_{ij}^{[F]}(x,y)$ and the Cartan nonlinear connection $^{[F]}\mathbf{N}_j^i(x,y)$ and any chosen d–connection structure (26) (see details on different type of d–connections in section 3). Additionally, the N–connection allows us to define an almost complex structure I on TM as follows

$$I(\delta_i) = -\partial/\partial y^i \text{ and } I\left(\partial/\partial y^i\right) = \delta_i$$

for which $I^2 = -1$.

The pair $\left(g^{[F]}, I\right)$ consisting from a Riemannian metric on a tangent bundle TM,

$$\mathbf{g}^{[F]} = g_{ij}^{[F]}(x,y) dx^i \otimes dx^j + g_{ij}^{[F]}(x,y) \delta y^i \otimes \delta y^j \tag{87}$$

and the almost complex structure I defines an almost Hermitian structure on $\widetilde{TM}$ associated to a 2–form

$$\theta = g_{ij}^{[F]}(x,y)\delta y^i \wedge dx^j.$$

This model of Finsler geometry is called almost Hermitian and denoted H^{2n} and it is proven [14] that is almost Kahlerian, i. e. the form θ is closed. The almost Kahlerian space $\mathbf{K}^{2n} = \left(\widetilde{TM}, \mathbf{g}^{[F]}, I\right)$ is also called the almost Kahlerian model of the Finsler space F^n.

On Finsler spaces (and their almost Kahlerian models), one distinguishes the almost Kahler linear connection of Finsler type, $\mathbf{D}^{[I]}$ on $\widetilde{TM}$ with the property that this covariant derivation preserves by parallelism the vertical distribution and is compatible with the almost Kahler structure $\left(\mathbf{g}^{[F]}, I\right)$, i.e.

$$\mathbf{D}_X^{[I]}\mathbf{g}^{[F]} = 0 \text{ and } \mathbf{D}_X^{[I]}\mathbf{I} = 0$$

for every d–vector field on $\widetilde{TM}$. This d–connection is defined by the data

$$^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma} = \left(^{[F]}\widehat{L}^i_{jk},\ ^{[F]}\widehat{L}^i_{jk},\ ^{[F]}\widehat{C}^i_{jk},\ ^{[F]}\widehat{C}^i_{jk}\right) \qquad (88)$$

with $^{[F]}\widehat{L}^i_{jk}$ and $^{[F]}\widehat{C}^i_{jk}$ computed by similar formulas in (56) by using $g_{ij}^{[F]}$ as in (85) and $^{[F]}N^i_j$ from (86).

We emphasize that a Finsler space $\mathbf{F}^n$ with a d–metric (87) and Cartan's N–connection structure (86), or the corresponding almost Hermitian (Kahler) model $\mathbf{H}^{2n}$, can be equivalently modeled on a space of dimension $2n$, $\mathbf{V}^{n+n}$, provided with an off–diagonal metric (34) and anholonomic frame structure with associated Cartan's nonlinear connection. Such anholonomic frame constructions are similar to modeling of the Einstein–Cartan geometry on (pseudo) Riemannian spaces where the torsion is considered as an effective tensor field. From this viewpoint a Finsler geometry is a Riemannian–Cartan geometry defined on a tangent bundle provided with a respective off–diagonal metric (and a related anholonomic frame structure with associated N–connection) and with additional prescriptions with respect to the type of linear connection chosen to be compatible with the metric and N–connection structures.

4.2.2 Finsler–Kaluza–Klein spaces

In Ref. [37] we defined a 'locally anisotropic' toroidal compactification of the 10 dimensional heterotic string sction [54]. We consider here the corresponding anholonomic frame transforms and off–diagonal metric ansatz. Let (n', m') be the (holonomic, anholonomic) dimensions of the compactified spacetime (as a particular case we can state $n' + m' = 4$, or any integers $n' + m' < 10$, for instance, for brane configurations). There are used such parametrizations of indices and of vierbeinds: Greek indices $\alpha, \beta, ...\mu...$ run values for a 10 dimensional spacetime and split as $\alpha = (\alpha', \widehat{\alpha}), \beta = \left(\beta', \widehat{\beta}\right), ...$ when primed indices $\alpha', \beta', ...\mu'...$ run values for compactified spacetime and split into h- and v–components like $\alpha' = (i', a'), \beta' = (j', b'), ...$; the frame coefficients are split as

$$e_\mu^{\ \underline{\mu}}(u) = \begin{pmatrix} e_{\alpha'}^{\ \underline{\alpha}'}(u^{\beta'}) & A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'})e_{\widehat{\alpha}}^{\ \underline{\widehat{\alpha}}}(u^{\beta'}) \\ 0 & e_{\widehat{\alpha}}^{\ \underline{\widehat{\alpha}}}(u^{\beta'}) \end{pmatrix} \qquad (89)$$

where $e_{\underline{\alpha'}}^{\ \alpha'}(u^{\beta'})$, in their turn, are taken in the form (16),

$$e_{\underline{\alpha'}}^{\ \alpha'}(u^{\beta'}) = \left(\begin{array}{cc} e_{i'}^{\ \underline{i'}}(x^{j'}, y^{a'}) & N_{i'}^{a'}(x^{j'}, y^{a'}) e_{a'}^{\ \underline{a'}}(x^{j'}, y^{a'}) \\ 0 & e_{a'}^{\ \underline{a'}}(x^{j'}, y^{a'}) \end{array} \right). \quad (90)$$

For the metric

$$\mathbf{g} = \underline{g}_{\alpha\beta} du^{\alpha} \otimes du^{\beta} \quad (91)$$

we have the recurrent ansatz

$$\underline{g}_{\alpha\beta} = \left[\begin{array}{cc} g_{\alpha'\beta'}(u^{\beta'}) + A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) A_{\beta'}^{\widehat{\beta}}(u^{\beta'}) h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'}) \\ h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) A_{\beta'}^{\widehat{\beta}}(u^{\beta'}) & h_{\widehat{\alpha}\widehat{\beta}}(u^{\beta'}) \end{array} \right], \quad (92)$$

where

$$g_{\alpha'\beta'} = \left[\begin{array}{cc} g_{i'j'}(u^{\beta'}) + N_{i'}^{a'}(u^{\beta'}) N_{j'}^{b'}(u^{\beta'}) h_{a'b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'}) N_{i'}^{a'}(u^{\beta'}) \\ h_{a'b'}(u^{\beta'}) N_{j'}^{b'}(u^{\beta'}) & h_{a'b'}(u^{\beta'}) \end{array} \right]. \quad (93)$$

After a toroidal compactification on $u^{\widehat{\alpha}}$ with gauge fields $A_{\alpha'}^{\widehat{\alpha}}(u^{\beta'})$, generated by the frame transform (89), we obtain a metric (91) like in the usual Kaluza–Klein theory (79) but containing the values $g_{\alpha'\beta'}(u^{\beta'})$, defined as in (93) (in a generic off–diagonal form similar to (34), labeled by primed indices), which can be induced as in Finsler geometry. This is possible if $g_{i'j'}(u^{\beta'}), h_{a'b'}(u^{\beta'}) \rightarrow g_{i'j'}^{[F]}(x', y')$ (see (85)) and $N_{i'}^{a'}(u^{\beta'}) \rightarrow N_{j'}^{[F]i'}(x', y')$ (see (86)) inducing a Finsler space with "primed" labels for objects. Such locally anisotropic spacetimes (in this case we emphasized the Finsler structures) can be generated anisotropic toroidal compactifications from different models of higher dimension of gravity (string, brane, or usual Kaluza–Klein theories). They define a mixed variant of Finsler and Kaluza–Klein spaces.

By using the recurent ansatz (92) and (93), we can generate both nontrivial nonmetricity and prescribed torsion structures adapted to a corresponding N–connection $N_{i'}^{a'}$. For instance, (after topological compactification on higher dimension) we can prescribe in the lower dimensional spacetime certain torsion fields $T_{\ i'j'}^{k'}$ and $T_{\ b'c'}^{a'}$ (they could have a particular relation to the so called B–fields in string theory, or connected to other models). The next steps are to compute $\tau_{\ i'j'}^{k'}$ and $\tau_{\ b'c'}^{a'}$ by using formulas (65) and define

$$^{[B\tau]}\mathbf{\Gamma}_{\ \alpha'\beta'}^{\gamma'} = \left(L_{j'k'}^{i'} = \widehat{L}_{\ j'k'}^{i'} + \tau_{\ j'k'}^{i'},\ L_{.b'k'}^{a'} = \frac{\partial N_{k'}^{a'}}{\partial y^{b'}}, C_{.j'a'}^{i'} = 0, C_{b'c'}^{a'} = \widehat{C}_{\ b'c'}^{a'} + \tau_{\ b'c'}^{a'} \right) \quad (94)$$

as in (64) (all formulas being with primed indices and $\widehat{L}_{\ j'k'}^{i'}$ and $\widehat{C}_{\ b'c'}^{a'}$ defined as in (56)). This way we can generate from Kaluza–Klein/ string theory a Berwald spacetime with nontrivial N–adapted nonmetricity

$$^{[B\tau]}\mathbf{Q}_{\alpha'\beta'\gamma'} = \ ^{[B\tau]}\mathbf{D}\mathbf{g}_{\beta'\gamma'} = \left(^{[B\tau]}Q_{c'i'j'},\ ^{[B\tau]}Q_{i'a'b'} \right)$$

and torsions $^{[B\tau]}\mathbf{T}_{\ \beta'\gamma'}^{\alpha'}$ with h- and v– irreducible components

$$\begin{aligned} T_{.j'k'}^{i'} &= -T_{k'j'}^{i'} = L_{j'k'}^{i'} - L_{k'j'}^{i'}, \quad T_{j'a'}^{i'} = -T_{a'j'}^{i'} = C_{.j'a'}^{i'},\ T_{.i'j'}^{a'} = \frac{\delta N_{i'}^{a'}}{\delta x^{j'}} - \frac{\delta N_{j'}^{a'}}{\delta x^{i'}}, \\ T_{.b'i'}^{a'} &= -T_{.i'b'}^{a'} = \frac{\partial N_{i'}^{a'}}{\partial y^{b'}} - L_{.b'j'}^{a'},\ T_{.b'c'}^{a'} = -T_{.c'b'}^{a'} = C_{b'c'}^{a'} - C_{c'b'}^{a'}. \end{aligned} \quad (95)$$

defined by the h- and v–coefficients of (94).

We conclude that if toroidal compactifications are locally anisotropic, defined by a chain of ansatz containing N–connection, the lower dimensional spacetime can be not only with torsion structure (like in low energy limit of string theory) but also with nonmetricity. The anholonomy induced by N–connection gives the possibility to define a more wide class of linear connections adapted to the h- and v–splitting.

4.2.3 Finsler–Riemann–Cartan spaces

Such spacetimes are modeled as Riemann–Cartan geometries on a tangent bundle TM when the metric and anholonomic frame structures distinguished to be of Finsler type (87). Both Finsler and Riemann–Cartan spaces possess nontrivial torsion structures (see section 2.4 for details on definition and computation torsions of locally anisotropic spaces and Refs. [38] for a review of the Einstein–Cartan gravity). The fundamental geometric objects defining Finsler–Riemann–Cartan spaces consists in the triple $\left(\mathbf{g}^{[F]}, \vartheta^{\alpha}_{[F]}, \mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}\right)$ where $\mathbf{g}^{[F]}$ is a d–metric (87), $\vartheta^{\alpha}_{[F]} = \left(dx^i, \delta y^j = dy^j + N^j_{[F]k}\left(x^l, y^s\right) dx^k\right)$ with $N^j_{[F]k}\left(x^l, y^s\right)$ of type (86) and $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$ is an arbitrary d–connection (26) on TM (we put the label [F] emphasizing that the N–connection is a Finsler type one). The torsion $\mathbf{T}^{\alpha}_{[F]}$ and curvature $\mathbf{R}^{\alpha}_{[F]\beta}$ d–forms are computed following respectively the formulas (41) and (42) but for $\vartheta^{\alpha}_{[F]}$ and $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$.

We can consider an inverse modeling of geometries when (roughly speaking) the Finsler configurations are 'hidden' in Riemann–Cartan spaces. They can be distinguished for arbitrary Riemann–Cartan manifolds V^{n+n} coventionally splitted into "horizontal" and "vertical" subspaces and provided with a metric ansatz of type (87) and with prescribed procedure of adapting the geometric objects to the Cartan N–connection $N^j_{[F]k}$. Of course, the torsion can not be an arbitrary one but admitting irreducible decompositions with respect to N–frames $\mathbf{e}^{[F]}_{\alpha}$ and N–coframes $\vartheta^{\alpha}_{[F]}$ (see, respectively, the formulas (21) and (22) when $N^a_i \rightarrow N^j_{[F]i}$). There were constructed and investigated different classes of exact solutions of the Einstein equations with anholonomic variables characterized by anholonomically induced torsions and modelling Finsler like geometries in (pseudo) Riemannian and Riemann–Cartan spaces (see Refs. [34, 35, 25]). All constructions from Finsler–Riemann–Cartan geometry reduce to Finsler–Riemann configurations (in general, we can see metrics of arbitrary signatures) if $\mathbf{\Gamma}^{\gamma}_{[F]\alpha\beta}$ is changed into the Levi–Civita metric connection defined with respect to anholonomic frames $\mathbf{e}_{\alpha}$ and coframes ϑ^{α} when the N–connection curvature Ω^i_{jk} and the anholonomically induced torsion vanish.

4.2.4 Teleparallel generalized Finsler geometry

In Refs. [55] the teleparallel Finsler connections, the Cartan–Einstein unification in the teleparallel approach and related moving frames with Finsler structures were investigated. In our analysis of teleparallel geometry we heavily use the results on N–connection geometry in order to illustrate how the teleparallel and metric affine gavity [9] can defined as to include generalized Finsler structures. For a general metric–affine spaces admitting N–connection structure N^a_i, the curvature $\mathbf{R}^{\alpha}_{.\beta\gamma\tau}$ of an arbitrary d–connection $\mathbf{\Gamma}^{\gamma}_{\alpha\beta} = \left(L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc}\right)$ splits into h– and v–irreversible components, $\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = (R^i{}_{hjk}, R^a{}_{bjk}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{bcd}),$

see (48). In order to include Finsler like metrics, we state that the N–connection curvature can be nontrivial $\Omega^a_{jk} \neq 0$, which is quite different from the condition imposed in section 4.1.3. The condition of vanishing of curvature for teleparallel spaces, see (80), is to be stated separately for every h- v–irreversible component,

$$R^i{}_{hjk} = 0, R^a{}_{bjk} = 0, P^i{}_{jka} = 0, P^c{}_{bka} = 0, S^i{}_{jbc} = 0, S^a{}_{bcd} = 0.$$

We can define certain types of teleparallel Berwald connections (see sections 3.4.1 and 3.4.2) with certain nontrivial components of nonmetricity d–field (63) if we modify the metric compatibility conditions (81) into a less strong one when

$$Q_{kij} = -D_k g_{ij} = 0 \text{ and } Q_{abc} = -D_a h_{bc} = 0$$

but with nontrivial components

$$\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij} = -D_c g_{ij},\ Q_{iab} = -D_i h_{ab})\,.$$

The class of teleparallel Finsler spaces is distinguished by Finsler N–connection and d–connection ${}^{[F]}\mathbf{N}^i_j(x,y)$ and ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} = \left({}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{L}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk},\ {}^{[F]}\widehat{C}^i_{jk}\right)$, see, respectively, (86) and (88) with vanishing d–curvature components,

$${}^{[F]}R^i{}_{hjk} = 0,\ {}^{[F]}P^i{}_{jka} = 0,\ {}^{[F]}S^i{}_{jbc} = 0.$$

We can generate teleparallel Finsler affine structures if it is not imposed the condition of vanishing of nonmetricity d–field. In this case, there are considered arbitrary d–connections $\mathbf{D}_\alpha$ that for the induced Finsler quadratic form (87) $\mathbf{g}^{[F]}$

$${}^{[F]}\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_\alpha \mathbf{g}^{[F]} \neq 0$$

but $\mathbf{R}^{\alpha}_{.\beta\gamma\tau}(\mathbf{D}) = 0$.

The teleparallel–Finsler configurations are contained as particular cases of Finsler–affine spaces, see section 4.2.4. For vielbein fields $\mathbf{e}_\alpha^{\ \underline{\alpha}}$ and their inverses $\mathbf{e}^\alpha_{\ \underline{\alpha}}$ related to the d–metric (87),

$$\mathbf{g}^{[F]}_{\alpha\beta} = \tilde{\mathbf{e}}_\alpha^{\ \underline{\alpha}}\tilde{\mathbf{e}}_\beta^{\ \underline{\beta}} g_{\underline{\alpha}\underline{\beta}}$$

we define the Weitzenbock–Finsler d–connection

$${}^{[WF]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \tilde{\mathbf{e}}^\alpha_{\ \underline{\alpha}}\delta_\gamma \tilde{\mathbf{e}}_\beta^{\ \underline{\alpha}} \tag{96}$$

where δ_γ are the elongated by ${}^{[F]}\mathbf{N}^i_j(x,y)$ partial derivatives (21). The torsion of ${}^{[WF]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ is defined

$${}^{[WF]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = {}^{[WF]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} - {}^{[WF]}\mathbf{\Gamma}^{\alpha}{}_{\gamma\beta} \tag{97}$$

containing h– and v–irreducible components being constructed from the components of a d–metric and N–adapted frames. We can express

$${}^{[WF]}\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$$

where $\mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$ is the Levi–Civita connection (53), $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} - \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma}$, and the contorsion tensor is

$$\mathbf{Z}_{\alpha\beta} = \mathbf{e}_\beta \rfloor\ {}^{[W]}\mathbf{T}_\alpha - \mathbf{e}_\alpha \rfloor\ {}^{[W]}\mathbf{T}_\beta + \frac{1}{2}\left(\mathbf{e}_\alpha \rfloor \mathbf{e}_\beta \rfloor\ {}^{[W]}\mathbf{T}_\gamma\right)\vartheta^\gamma + \left(\mathbf{e}_\alpha \rfloor \mathbf{Q}_{\beta\gamma}\right)\vartheta^\gamma - \left(\mathbf{e}_\beta \rfloor \mathbf{Q}_{\alpha\gamma}\right)\vartheta^\gamma + \frac{1}{2}\mathbf{Q}_{\alpha\beta}.$$

In the non-Berwald standard approaches to the Finsler–teleparallel gravity it is considered that $\mathbf{Q}_{\alpha\beta} = 0$.

4.2.5 Cartan geometry

The theory of Cartan spaces (see, for instance, [16, 56]) can be reformulated as a dual to Finsler geometry [58] (see details and references in [20]). The Cartan space is constructed on a cotangent bundle T^*M similarly to the Finsler space on the tangent bundle TM.

Consider a real smooth manifold M, the cotangent bundle (T^*M, π^*, M) and the manifold $\widetilde{T^*M} = T^*M \backslash \{0\}$.

Definition 4.1. *A Cartan space is a pair $C^n = (M, K(x,p))$ such that $K : T^*M \to \mathbb{R}$ is a scalar function satisfying the following conditions:*

1. *K is a differentiable function on the manifold $\widetilde{T^*M} = T^*M \backslash \{0\}$ and continuous on the null section of the projection $\pi^* : T^*M \to M$;*
2. *K is a positive function, homogeneous on the fibers of the T^*M, i. e. $K(x, \lambda p) = \lambda F(x,p), \lambda \in \mathbb{R}$;*
3. *The Hessian of K^2 with elements*

$$\check{g}^{ij}_{[K]}(x,p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_j} \tag{98}$$

   is positively defined on $\widetilde{T^*M}$.

The function $K(x,y)$ and $\check{g}^{ij}(x,p)$ are called respectively the fundamental function and the fundamental (or metric) tensor of the Cartan space C^n. We use symbols like "$\check{g}$" as to emphasize that the geometrical objects are defined on a dual space.

One considers "anisotropic" (depending on directions, momenta, p_i) Christoffel symbols. For simplicity, we write the inverse to (98) as $g^{(K)}_{ij} = \check{g}_{ij}$ and introduce the coefficients

$$\check{\gamma}^i{}_{jk}(x,p) = \frac{1}{2}\check{g}^{ir}\left(\frac{\partial \check{g}_{rk}}{\partial x^j} + \frac{\partial \check{g}_{jr}}{\partial x^k} - \frac{\partial \check{g}_{jk}}{\partial x^r}\right),$$

defining the canonical N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}\}$,

$$\check{N}^{[K]}_{ij} = \check{\gamma}^k{}_{ij} p_k - \frac{1}{2}\gamma^k{}_{nl} p_k p^l \check{\partial}^n \check{g}_{ij} \tag{99}$$

where $\check{\partial}^n = \partial/\partial p_n$. The N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}\}$ can be used for definition of an almost complex structure like in (87) and introducing on T^*M a d–metric

$$\check{\mathbf{G}}_{[k]} = \check{g}_{ij}(x,p)dx^i \otimes dx^j + \check{g}^{ij}(x,p)\delta p_i \otimes \delta p_j, \tag{100}$$

with $\check{g}^{ij}(x,p)$ taken as (98).

Using the canonical N–connection (99) and Finsler metric tensor (98) (or, equivalently, the d–metric (100)), we can define the canonical d–connection $\check{\mathbf{D}} = \{\check{\mathbf{\Gamma}}\left(\check{N}_{[k]}\right)\}$

$$\check{\mathbf{\Gamma}}\left(\check{N}_{[k]}\right) = \check{\Gamma}^{\alpha}_{[k]\beta\gamma} = \left(\check{H}^i_{[k]\ jk}, \check{H}^i_{[k]\ jk}, \check{C}_{[k]\ i}^{\ jk}, \check{C}_{[k]\ i}^{\ jk}\right)$$

with the coefficients computed

$$\check{H}^{i}_{[k]\ jk} = \frac{1}{2}\check{g}^{ir}\left(\check{\delta}_j \check{g}_{rk} + \check{\delta}_k \check{g}_{jr} - \check{\delta}_r \check{g}_{jk}\right),\ \check{C}_{[k]\ i}^{\ jk} = \check{g}_{is}\check{\partial}^s \check{g}^{jk}.$$

The d–connection $\check{\mathbf{\Gamma}}\left(\check{N}_{(k)}\right)$ satisfies the metricity conditions both for the horizontal and vertical components, i. e. $\check{\mathbf{D}}_\alpha \check{\mathbf{g}}_{\beta\gamma} = 0.$

The d–torsions (95) and d–curvatures (48) are computed like in Finsler geometry but starting from the coefficients in (99) and (100), when the indices $a, b, c...$ run the same values as indices $i, j, k, ...$ and the geometrical objects are modeled as on the dual tangent bundle. It should be emphasized that in this case all values $\check{g}_{ij}$, $\check{\Gamma}^{\alpha}_{[k]\beta\gamma}$ and $\check{R}^{\cdot\alpha}_{[k]\beta\gamma\delta}$ are defined by a fundamental function $K(x,p)$.

In general, we can consider that a Cartan space is provided with a metric $\check{g}^{ij} = \partial^2 K^2 / 2\partial p_i \partial p_j$, but the N–connection and d–connection could be defined in a different manner, even not be determined by K. If a Cartan space is modeled in a metric–affine space V^{n+n}, with local coordinates $\left(x^i, y^k\right)$, we have to define a procedure of dualization of vertical coordinates, $y^k \to p_k$.

4.3 Generalized Lagrange and Hamilton geometries

The notion of Finsler spaces was extended by J. Kern [57] and R. Miron [60]. It is was elaborated in vector bundle spaces in Refs. [14] and generalized to superspaces [23]. We illustrate how such geometries can be modeled on a space $\mathbf{V}^{n+n}$ provided with N–connection structure.

4.3.1 Lagrange geometry and generalizations

The idea of generalization of the Finsler geometry was to consider instead of the homogeneous fundamental function $F(x,y)$ in a Finsler space a more general one, a Lagrangian $L(x,y)$, defined as a differentiable mapping $L : (x,y) \in TV^{n+n} \to L(x,y) \in \mathbb{R}$, of class C^∞ on manifold $\widetilde{TV}^{n+n}$ and continuous on the null section $0 : V^n \to \widetilde{TV}^{n+n}$ of the projection $\pi : \widetilde{TV}^{n+n} \to V^n$. A Lagrangian is regular if it is differentiable and the Hessian

$$g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j} \tag{101}$$

is of rank n on V^n.

Definition 4.2. *A Lagrange space is a pair $\mathbf{L}^n = (V^n, L(x,y))$ where V^n is a smooth real n–dimensional manifold provided with regular Lagrangian $L(x,y)$ structure $L : TV^n \to \mathbb{R}$ for which $g_{ij}(x,y)$ from (101) has a constant signature over the manifold $\widetilde{TV}^{n+n}$.*

The fundamental Lagrange function $L(x,y)$ defines a canonical N–connection

$$^{[cL]}N^i_{\ j} = \frac{1}{2}\frac{\partial}{\partial y^j}\left[g^{ik}\left(\frac{\partial^2 L^2}{\partial y^k \partial y^h}y^h - \frac{\partial L}{\partial x^k}\right)\right] \tag{102}$$

as well a d-metric

$$\mathbf{g}_{[L]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{103}$$

with $g_{ij}(x,y)$ taken as (101). As well we can introduce an almost Kählerian structure and an almost Hermitian model of $\mathbf{L}^n$, denoted as $\mathbf{H}^{2n}$ as in the case of Finsler spaces but with a proper fundamental Lagrange function and metric tensor g_{ij}. The canonical metric d–connection $\widehat{\mathbf{D}}_{[L]}$ is defined by the coefficients

$$^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left({}^{[L]}\widehat{L}^{i}{}_{jk},\ {}^{[L]}\widehat{L}^{i}{}_{jk},\ {}^{[L]}\widehat{C}^{i}{}_{jk},\ {}^{[L]}\widehat{C}^{i}{}_{jk} \right) \tag{104}$$

computed for $N^i_{[cL]\ j}$ and by respective formulas (56) with $h_{ab} \to g^{[L]}_{ij}$ and $\widehat{C}^a_{bc} \to \widehat{C}^i_{ij}$. The d–torsions (95) and d–curvatures (48) are determined, in this case, by ${}^{[L]}\widehat{L}^{i}{}_{jk}$ and ${}^{[L]}\widehat{C}^{i}{}_{jk}$. We also note that instead of ${}^{[cL]}N^i{}_j$ and ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ we can consider on a L^n–space different N–connections $N^i{}_j$, d–connections $\mathbf{\Gamma}^{\alpha}{}_{\beta\gamma}$ which are not defined only by $L(x,y)$ and $g^{[L]}_{ij}$ but can be metric, or non–metric with respect to the Lagrange metric.

The next step of generalization [60] is to consider an arbitrary metric $g_{ij}(x,y)$ on $\mathbf{TV}^{n+n}$ (we use boldface symbols in order to emphasize that the space is enabled with N–connection structure) instead of (101) which is the second derivative of "anisotropic" coordinates y^i of a Lagrangian.

Definition 4.3. *A generalized Lagrange space is a pair* $\mathbf{GL}^n = (V^n, g_{ij}(x,y))$ *where* $g_{ij}(x,y)$ *is a covariant, symmetric and N–adapted d–tensor field of rank* n *and of constant signature on* $\widetilde{TV}^{n+n}$.

One can consider different classes of N– and d–connections on TV^{n+n}, which are compatible (metric) or non compatible with (103) for arbitrary $g_{ij}(x,y)$ and arbitrary d–metric

$$\mathbf{g}_{[gL]} = g_{ij}(x,y)dx^i \otimes dx^j + g_{ij}(x,y)\delta y^i \otimes \delta y^j, \tag{105}$$

We can apply all formulas for d–connections, N–curvatures, d–torsions and d–curvatures as in sections 2.3 and 2.4 but reconsidering them on $\mathbf{TV}^{n+n}$, by changing

$$h_{ab} \to g_{ij}(x,y), \widehat{C}^a_{bc} \to \widehat{C}^i_{ij} \text{ and } N^a_i \to N^k{}_i.$$

Prescribed torsions $T^i{}_{jk}$ and $S^i{}_{jk}$ can be introduced on $\mathbf{GL}^n$ by using the d–connection

$$\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left(\widehat{L}^i_{[gL]\ jk} + \tau^i{}_{jk}, \widehat{L}^i_{[gL]\ jk} + \tau^i{}_{jk}, \widehat{C}^i_{[gL]\ jk} + \sigma^i{}_{jk}, \widehat{C}^i_{[gL]\ jk} + \sigma^i{}_{jk} \right) \tag{106}$$

with

$$\tau^i{}_{jk} = \frac{1}{2} g^{il}\left(g_{kh}T^h_{.lj} + g_{jh}T^h_{.lk} - g_{lh}T^h{}_{jk}\right) \text{ and } \sigma^i{}_{jk} = \frac{1}{2} g^{il}\left(g_{kh}S^h_{.lj} + g_{jh}S^h_{.lk} - g_{lh}S^h{}_{jk}\right)$$

like we have performed for the Berwald connections (64) with (65) and (94) but in our case

$$^{[aL]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = \left(\widehat{L}^i_{[gL]\ jk}, \widehat{L}^i_{[gL]\ jk}, \widehat{C}^i_{[gL]\ jk}, \widehat{C}^i_{[gL]\ jk} \right) \tag{107}$$

is metric compatible being modeled like on a tangent bundle and with the coefficients computed as in (56) with $h_{ab} \to g^{[L]}_{ij}$ and $\widehat{C}^a_{bc} \to \widehat{C}^i_{ij}$, by using the d–metric $\mathbf{G}_{[gL]}$ (105). The connection (106) is a Riemann–Cartan one modeled on effective tangent bundle provided with N–connection structure.

4.3.2 Hamilton geometry and generalizations

The geometry of Hamilton spaces was defined and investigated by R. Miron in the papers [59] (see details and additional references in [20]). It was developed on the cotangent bundle as a dual geometry to the geometry of Lagrange spaces. Here we consider their modeling on couples of spaces $(V^n, \ ^*V^n)$, or cotangent bundle T^*M, where $^*V^n$ is considered as a 'dual' manifold defined by local coordinates satisfying a duality condition with respect to coordinates on V^n. We start with the definition of generalized Hamilton spaces and then consider the particular cases.

Definition 4.4. *A generalized Hamilton space is a pair* $\mathbf{GH}^n = (V^n, \check{g}^{ij}(x,p))$ *where* V^n *is a real* n*–dimensional manifold and* $\check{g}^{ij}(x,p)$ *is a contravariant, symmetric, nondegenerate of rank* n *and of constant signature on* $\widetilde{T^*V}^{n+n}$.

The value $\check{g}^{ij}(x,p)$ is called the fundamental (or metric) tensor of the space $\mathbf{GH}^n$. One can define such values for every paracompact manifold V^n. In general, a N–connection on $\mathbf{GH}^n$ is not determined by $\check{g}^{ij}$. Therefore we can consider an arbitrary N–connection $\check{\mathbf{N}} = \{\check{N}_{ij}(x,p)\}$ and define on T^*V^{n+n} a d–metric similarly to (33) and/or (103)

$$\mathbf{G}_{[gH]}^{\smile} = \check{g}_{\alpha\beta}(\check{u})\,\check{\delta}^{\alpha}\otimes\check{\delta}^{\beta} = \check{g}_{ij}(\check{u})\,d^i\otimes d^j + \check{g}^{ij}(\check{u})\,\check{\delta}_i\otimes\check{\delta}_j, \tag{108}$$

The N–coefficients $\check{N}_{ij}(x,p)$ and the d–metric structure (108) define an almost Kähler model of generalized Hamilton spaces provided with canonical d–connections, d–torsions and d-curvatures (see respectively the formulas d–torsions (95) and d–curvatures (48) with the fiber coefficients redefined for the cotangent bundle $T^*V^{\ n+n}$).

A generalized Hamilton space $\mathbf{GH}^n$ is called reducible to a Hamilton one if there exists a Hamilton function $H(x,p)$ on $T^*V^{\ n+n}$ such that

$$\check{g}^{ij}_{[H]}(x,p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}. \tag{109}$$

Definition 4.5. *A Hamilton space is a pair* $\mathbf{H}^n = (V^n, H(x,p))$ *such that* $H : T^*V^n \to \mathbb{R}$ *is a scalar function which satisfy the following conditions:*

1. *H is a differentiable function on the manifold* $\widetilde{T^*V}^{n+n} = T^*V^{n+n}\backslash\{0\}$ *and continuous on the null section of the projection* $\pi^* : T^*V^{n+n} \to V^n$;

2. *The Hessian of H with elements (109) is positively defined on* $\widetilde{T^*V}^{n+n}$ *and* $\check{g}^{ij}(x,p)$ *is nondegenerate matrix of rank n and of constant signature.*

For Hamilton spaces, the canonical N–connection (defined by H and its Hessian) is introduced as

$$^{[H]}\check{N}_{ij} = \frac{1}{4}\{\check{g}_{ij}, H\} - \frac{1}{2}\left(\check{g}_{ik}\frac{\partial^2 H}{\partial p_k \partial x^j} + \check{g}_{jk}\frac{\partial^2 H}{\partial p_k \partial x^i}\right), \tag{110}$$

where the Poisson brackets, for arbitrary functions f and g on T^*V^{n+n}, act as

$$\{f, g\} = \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i} - \frac{\partial g}{\partial p_i}\frac{\partial p}{\partial x^i}.$$

The canonical metric d–connection ${}^{[H]}\widehat{\mathbf{D}}$ is defined by the coefficients

$$ {}^{[H]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma} = \left({}^{[c]}\check{H}^{i}_{\ jk},\ {}^{[c]}\check{H}^{i}_{\ jk},\ {}^{[c]}\check{C}^{i}_{\ jk},\ {}^{[c]}\check{C}^{i}_{\ jk} \right) $$

computed for ${}^{[H]}\check{N}_{ij}$ and by respective formulas (56) with $g_{ij} \rightarrow \check{g}_{ij}(\check{u})$, $h_{ab} \rightarrow \check{g}^{ij}$ and $\widehat{L}^{i}_{\ jk} \rightarrow {}^{[c]}\widehat{H}^{i}_{\ jk},\ \widehat{C}^{a}_{bc} \rightarrow {}^{[c]}\check{C}_{\ i}^{\ \ jk}$ when

$$ {}^{[c]}\check{H}^{i}_{\ jk} = \frac{1}{2}\check{g}^{is}\left(\check{\delta}_{j}\check{g}_{sk} + \check{\delta}_{k}\check{g}_{js} - \check{\delta}_{s}\check{g}_{jk}\right) \text{ and } {}^{[c]}\check{C}_{\ i}^{\ \ jk} = -\frac{1}{2}\check{g}_{is}\check{\partial}^{j}\check{g}^{sk}. $$

In result, we can compute the d–torsions and d–curvatures like on Lagrange or on Cartan spaces. On Hamilton spaces all such objects are defined by the Hamilton function $H(x,p)$ and indices have to be reconsidered for co–fibers of the cotangent bundle.

We note that there were elaborated various type of higher order generalizations (on the higher order tangent and contangent bundles) of the Finsler–Cartan and Lagrange–Hamilton geometry [21] and on higher order supersymmetric (co) vector bundles in Ref. [23]. We can generalize the d–connection ${}^{[H]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ to any d–connection in $\mathbf{H}^{n}$ with prescribed torsions, like we have done in previous section for Lagrange spaces, see (106). This type of Riemann–Cartan geometry is modeled like on a dual tangent bundle by a Hamilton metric structure (109), N–connection ${}^{[H]}\check{N}_{ij}$, and d–connection coefficients ${}^{[c]}\check{H}^{i}_{\ jk}$ and ${}^{[c]}\check{C}_{\ i}^{\ \ jk}$.

4.4 Nonmetricity and generalized Finsler–affine spaces

The generalized Lagrange and Finsler geometry may be defined on tangent bundles by using d–connections and d–metrics satisfying metric compatibility conditions [14]. Nonmetricity components can be induced if Berwald type d–connections are introduced into consideration on different type of manifolds provided with N–connection structure, see formulas (62), (64), (70) and (94).

We define such spaces as generalized Finsler spaces with nonmetricity.

Definition 4.6. *A generalized Lagrange–affine space* $\mathbf{GLa}^{n} = \left(V^{n}, g_{ij}(x,y),\ {}^{[a]}\mathbf{\Gamma}^{\alpha}_{\ \beta}\right)$ *is defined on manifold* $\mathbf{TV}^{n+n}$*, provided with an arbitrary nontrivial N–connection structure* $\mathbf{N} = \{N^{i}_{j}\}$*, as a general Lagrange space* $\mathbf{GL}^{n} = (V^{n}, g_{ij}(x,y))$ *(see Definition 4.3) enabled with a d–connection structure* ${}^{[a]}\mathbf{\Gamma}^{\gamma}_{\ \ \alpha} = {}^{[a]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$$ {}^{[a]}\mathbf{\Gamma}^{\alpha}_{\ \beta} = {}^{[aL]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta} + {}^{[a]}\mathbf{Z}^{\alpha}_{\ \ \beta}, \tag{111} $$

where ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta}$ *is the canonical generalized Lagrange d–connection (107) and*

$$ {}^{[a]}\mathbf{Z}_{\ \alpha\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta}. $$

The d–metric structure on $\mathbf{GLa}^{n}$ is stated by an arbitrary N–adapted form (33) but on $\mathbf{TV}^{n+n}$,

$$ \mathbf{g}_{[a]} = g_{ij}(x,y)dx^{i}\otimes dx^{j} + g_{ij}(x,y)\delta y^{i}\otimes\delta y^{j}. \tag{112} $$

The torsions and curvatures on $\mathbf{GLa}^n$ are computed by using formulas (41) and (42) with $\mathbf{\Gamma}^{\gamma}{}_{\beta} \rightarrow {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}$,

$$^{[a]}\mathbf{T}^{\alpha} \doteqdot {}^{[a]}\mathbf{D}\vartheta^{\alpha} = \delta\vartheta^{\alpha} + {}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge \vartheta^{\beta} \tag{113}$$

and

$$^{[a]}\mathbf{R}^{\alpha}{}_{\beta} \doteqdot {}^{[a]}\mathbf{D}({}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}) = \delta({}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}) - {}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\beta} \wedge {}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\gamma}. \tag{114}$$

Modeling in V^{n+n}, with local coordinates $u^{\alpha} = \left(x^i, y^k\right)$, a tangent bundle structure, we redefine the operators (22) and (21) respectively as

$$\mathbf{e}_{\alpha} \doteqdot \delta_{\alpha} = \left(\delta_i, \tilde{\partial}_k\right) \equiv \frac{\delta}{\delta u^{\alpha}} = \left(\frac{\delta}{\delta x^i} = \partial_i - N_i^a\left(u\right)\partial_a, \frac{\partial}{\partial y^k}\right) \tag{115}$$

and the N–elongated differentials (in brief, N–differentials)

$$\vartheta^{\beta} \doteqdot \delta^{\ \beta} = \left(d^i, \tilde{\delta}^k\right) \equiv \delta u^{\alpha} = \left(\delta x^i = dx^i, \delta y^k = dy^k + N_i^k\left(u\right)dx^i\right) \tag{116}$$

where Greek indices run the same values, $i, j, ... = 1, 2, ...n$ (we shall use the symbol "$\sim$" if one would be necessary to distinguish operators and coordinates defined on h– and v–subspaces).

Let us define the h– and v–irreducible components of the d–connection ${}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta}$ like in (27) and (28),

$$^{[a]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} = \left({}^{[L]}\widehat{L}^i{}_{jk} + z^i{}_{jk}, {}^{[L]}\widehat{L}^i{}_{jk} + z^i{}_{jk}, {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}, {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}\right)$$

with the distorsion d–tensor

$$^{[a]}\ \mathbf{Z}^{\alpha}{}_{\beta} = \left(z^i{}_{jk},\ z^i{}_{jk}, c^i{}_{jk}, c^i{}_{jk}\right)$$

defined as on a tangent bundle

$$\begin{aligned}
{}^{[a]}L^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\delta_k}\delta_j\right)\rfloor\delta^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\delta_k}\delta_j + {}^{[a]}\ \mathbf{Z}_{\delta_k}\delta_j\right)\rfloor\delta^i = {}^{[L]}\widehat{L}^i{}_{jk} + z^i{}_{jk}, \\
{}^{[a]}\tilde{L}^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\delta_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \left({}^{[L]}\widehat{\mathbf{D}}_k\tilde{\partial}_j + {}^{[a]}\ \mathbf{Z}_k\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = {}^{[L]}\widehat{L}^i{}_{jk} + z^i{}_{jk}, \\
{}^{[a]}C^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\tilde{\partial}_k}\delta_j\right)\rfloor\delta^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\tilde{\partial}_k}\delta_j + {}^{[a]}\ \mathbf{Z}_{\tilde{\partial}_k}\delta_j\right)\rfloor\delta^i = {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk}, \\
{}^{[a]}\tilde{C}^i_{jk} &= \left({}^{[a]}\mathbf{D}_{\tilde{\partial}_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = \left({}^{[L]}\widehat{\mathbf{D}}_{\tilde{\partial}_k}\tilde{\partial}_j + {}^{[a]}\ \mathbf{Z}_{\tilde{\partial}_k}\tilde{\partial}_j\right)\rfloor\tilde{\partial}^i = {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk},
\end{aligned}$$

where for 'lifts' from the h–subspace to the v–subspace we consider that ${}^{[a]}L^i_{jk} = {}^{[a]}\tilde{L}^i_{jk}$ and ${}^{[a]}C^i_{jk} = {}^{[a]}\tilde{C}^i_{jk}$. As a consequence, on spaces with modeled tangent space structure, the d–connections are distinguished as $\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \left(L^i_{jk}, C^i_{jk}\right)$.

Theorem 4.1. *The torsion* ${}^{[a]}\mathbf{T}^{\alpha}$ *(113) of a d–connection* ${}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \left({}^{[a]}L^i_{jk}, {}^{[a]}C^i_{jc}\right)$ *(111) has as irreducible h- v–components,* ${}^{[a]}\mathbf{T}^{\alpha} = \left(T^i_{jk}, \tilde{T}^i_{jk}\right)$, *the d–torsions*

$$\begin{aligned}
T^i_{.jk} &= -T^i_{kj} = {}^{[L]}\widehat{L}^i{}_{jk} + z^i{}_{jk} - {}^{[L]}\widehat{L}^i{}_{kj} - z^i{}_{kj}, \\
\tilde{T}^i_{jk} &= -\tilde{T}^i_{kj} = {}^{[L]}\widehat{C}^i{}_{jk} + c^i{}_{jk} - {}^{[L]}\widehat{C}^i{}_{kj} - c^i{}_{kj}.
\end{aligned} \tag{117}$$

The proof of this Theorem consists from a standard calculus for metric–affine spaces of ${}^{[a]}\mathbf{T}^{\alpha}$ [4] but with N–adapted frames. We note that in $z^{i}{}_{jk}$ and $c^{i}{}_{kj}$ it is possible to include any prescribed values of the d–torsions.

Theorem 4.2. *The curvature* ${}^{[a]}\mathbf{R}^{\alpha}{}_{\beta}$ *(114) of a d–connection* ${}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = \left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ *(111) has as irreducible h- v–components,* ${}^{[a]}\mathbf{R}^{\alpha}_{.\beta\gamma\tau} = \{\ {}^{[a]}R^{i}{}_{hjk},\ {}^{[a]}P^{i}{}_{jka},\ {}^{[a]}S^{i}{}_{jbc}\}$ *the d–curvatures*

$$
\begin{aligned}
{}^{[a]}R^{i}{}_{hjk} &= \frac{\delta\ {}^{[a]}L^{i}_{.hj}}{\delta x^{h}} - \frac{\delta\ {}^{[a]}L^{i}_{.hk}}{\delta x^{j}} + {}^{[a]}L^{m}_{.hj}\ {}^{[a]}L^{i}_{mk} - {}^{[a]}L^{m}_{.hk}\ {}^{[a]}L^{i}_{mj} - {}^{[a]}C^{i}_{.ho}\Omega^{o}_{.jk}, \\
{}^{[a]}P^{i}{}_{jks} &= \frac{\partial\ {}^{[a]}L^{i}_{.jk}}{\partial y^{s}} - \left(\frac{\partial\ {}^{[a]}C^{i}_{.js}}{\partial x^{k}} + {}^{[a]}L^{i}_{.lk}\ {}^{[a]}C^{l}_{.js} - {}^{[a]}L^{l}_{.jk}\ {}^{[a]}C^{i}_{.ls} - {}^{[a]}L^{p}_{.sk}\ {}^{[a]}C^{i}_{.jp} \right) \\
&\quad + {}^{[a]}C^{i}_{.jp}\ {}^{[a]}P^{p}_{.ks}, \\
{}^{[a]}S^{i}{}_{jlm} &= \frac{\partial\ {}^{[a]}C^{i}_{.jl}}{\partial y^{m}} - \frac{\partial\ {}^{[a]}C^{i}_{.jm}}{\partial y^{l}} + {}^{[a]}C^{h}_{.jl}\ {}^{[a]}C^{i}_{.hm} - {}^{[a]}C^{h}_{.jm}\ {}^{[a]}C^{i}_{hl},
\end{aligned}
$$

where ${}^{[a]}L^{m}_{.hk} = {}^{[L]}\widehat{L}^{i}{}_{jk} + z^{i}{}_{jk}$, ${}^{[a]}C^{i}_{.jk} = {}^{[L]}\widehat{C}^{i}{}_{jk} + c^{i}{}_{jk}$, $\Omega^{o}_{.jk} = \delta_{j}N^{o}_{i} - \delta_{i}N^{o}_{j}$ *and* ${}^{[a]}P^{p}_{.ks} = \partial N^{p}_{i}/\partial y^{s} - {}^{[a]}L^{p}_{.ks}$.

The proof consists from a straightforward calculus.

Remark 4.1. *As a particular case of* $\mathbf{GLa}^{n}$*, we can define a Lagrange–affine space* $\mathbf{La}^{n} = \left(V^{n}, g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$*, provided with a Lagrange quadratic form* $g^{[L]}_{ij}(x,y)$ *(101) inducing the canonical N–connection structure* ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^{i}_{j}\}$ *(102) as in a Lagrange space* $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ *(see Definition 4.2)) but with a d–connection structure* ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$$
{}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta},
$$

where ${}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta}$ *is the canonical Lagrange d–connection (104),*

$$
{}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta},
$$

and the (co) frames $\mathbf{e}_{\beta}$ *and* ϑ^{γ} *are respectively constructed as in (21) and (22) by using* ${}^{[cL]}N^{i}_{j}$.

Remark 4.2. *The Finsler–affine spaces* $\mathbf{Fa}^{n} = \left(V^{n}, F(x,y),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ *can be introduced by further restrictions of* $\mathbf{La}^{n}$ *to a quadratic form* $g^{[F]}_{ij}$ *(85) constructed from a Finsler metric* $F(x^{i}, y^{j})$ *inducing the canonical N–connection structure* ${}^{[F]}\mathbf{N} = \{\ {}^{[F]}N^{i}_{j}\}$ *(86) as in a Finsler space* $\mathbf{F}^{n} = (V^{n}, F(x,y))$ *but with a d–connection structure* ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ *distorted by arbitrary torsion,* $\mathbf{T}_{\beta}$*, and nonmetricity,* $\mathbf{Q}_{\beta\gamma}$*, d–fields,*

$$
{}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta},
$$

where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta}$ *is the canonical Finsler d–connection (88),*

$$^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta} = \mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\alpha} - \mathbf{e}_{\alpha}\rfloor\ \mathbf{T}_{\beta} + \frac{1}{2}\left(\mathbf{e}_{\alpha}\rfloor\mathbf{e}_{\beta}\rfloor\ \mathbf{T}_{\gamma}\right)\vartheta^{\gamma} + \left(\mathbf{e}_{\alpha}\rfloor\ \mathbf{Q}_{\beta\gamma}\right)\vartheta^{\gamma} - \left(\mathbf{e}_{\beta}\rfloor\ \mathbf{Q}_{\alpha\gamma}\right)\vartheta^{\gamma} + \frac{1}{2}\ \mathbf{Q}_{\alpha\beta},$$

and the (co) frames $\mathbf{e}_{\beta}$ *and* ϑ^{γ} *are respectively constructed as in (21) and (22) by using* ${}^{[F]}N_j^i$.

Remark 4.3. *By similar geometric constructions (see Remarks 4.1 and 4.2) on spaces modeling cotangent bundles, we can define generalized Hamilton–affine spaces* $\mathbf{GHa}^n = \left(V^n, \check{g}^{ij}(x,p),\ {}^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ *and theirs restrictions to Hamilton–affine* $\mathbf{Ha}^n = (V^n, \check{g}^{ij}_{[H]}(x,p), {}^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta})$ *and Cartan–affine spaces* $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p),\ {}^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ *(see sections 4.3.2 and 4.2.5) as to contain distorsions induced by nonmetricity* $\check{\mathbf{Q}}_{\alpha\gamma}$*. The geometric objects have to be adapted to the corresponding N–connection and d–metric/ quadratic form structures (arbitrary* $\check{N}_{ij}(x,p)$ *and d–metric (108),* ${}^{[H]}\check{N}_{ij}(x,p)$ *(110) and quadratic form* $\check{g}^{ij}_{[H]}$ *(109) and* $\check{N}^{[K]}_{ij}$ *(99) and* $\check{g}^{ij}_{[K]}$ *(98).*

Finally, in this section, we note that Theorems 4.1 and 4.2 can be reformulated in the forms stating procedures of computing d–torsions and d–curvatures on every type of spaces with nonmetricity and local anisotropy by adapting the abstract symbol and/or coordinate calculations with respect to corresponding N–connection, d–metric and canonical d–connection structures.

5 Conclusions

The method of moving anholonomic frames with associated nonlinear connection (N–connection) structure elaborated in this work on metric–affine spaces provides a general framework to deal with any possible model of locally isotropic and/or anisotropic interactions and geometries defined effectively in the presence of generic off–diagonal metric and linear connection configurations, in general, subjected to certain anholonomic constraints. As it has been pointed out, the metric–affine gravity (MAG) contains various types of generalized Finsler–Lagrange–Hamilton–Cartan geometries which can be distinguished by a corresponding N–connection structure and metric and linear connection adapted to the N–connection structure.

As far as the anholonomic frames, nonmetricity and torsion are considered as fundamental quantities, all mentioned geometries can be included into a unique scheme which can be developed on arbitrary manifolds, vector and tangent bundles and their dual bundles (co-bundles) or restricted to Riemann–Cartan and (pseudo) Riemannian spaces. We observe that a generic off–diagonal metric (which can not be diagonalized by any coordinate transform) defining a (pseudo) Riemannian space induces alternatively various type of Riemann–Cartan and Finsler like configuratons modeled by respective frame structures. The constructions are generalized if the linear connection structures are not constrained to metricity conditions. One can regard this as extensions to metric–affine spaces provided with N–connection structure modeling also bundle structures and generalized noncommutative symmetries of metrics and anholonomic frames.

In this paper we have studied the general properties of metric–affine spaces provided with N–connection structure. We formulated and proved the main theorems concerning general metric and nonlinear and linear connections in MAG. There were stated the criteria when the spaces with local isotropy and/or local anisotropy can be modeled in metric–affine spaces and on vector/ tangent bundles. We elaborated the concept of generalized Finsler–affine geometry as a sinthesis of metric–affine (with nontrivial torsion and nonmetricity) and Finsler like configurations (with nontrivial N–connection structure and locally anisotropic metrics and connections).

In a general sense, we note that the generalized Finsler–affine geometries are contained as anhlonomic and noncommutative configurations in extra dimension gravity models (string and brane models and certain limits to the Einstein and gauge gravity defined by off–diagonal metrics and anholonomic constraints). We would like to stress that the N–connection formalism developed for the metric–affine spaces relates the bulk geometry in string and/or MAG to gauge theories in vector/tangent bundles and to various type of non–Riemannian gravity models.

The approach presented here could be advantageous in a triple sense. First, it provides a uniform treatment of all metric and connection geometries, in general, with vector/tangent bundle structures which arise in various type of string and brane gravity models. Second, it defines a complete classification of the generalized Finsler–affine geometries stated in Tables 1-11 from the Appendix. Third, it states a new geometric method of constructing exact solutions with generic off–diagonal metric ansatz, torsions and nonmetricity, depending on 2–5 variables, in string and metric–affine gravity, with limits to the Einstein gravity, see Refs. [33].

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A Classification of Generalized Finsler–Affine Spaces

We outline and give a brief characterization of the main classes of generalized Finsler–affine spaces (see Tables 1–11). A unified approach to such very different locally isotropic and anisotropic geometries, defined in the framework of the metric–affine geometry, can be elaborated only by introducing the concept on N–connection (see Definition 2.10).

The N–connection curvature is computed following the formula $\Omega^a_{ij} = \delta_{[i} N^a_{j]}$, see (20), for any N–connection N^a_i. A d–connection $\mathbf{D} = [\mathbf{\Gamma}^\alpha_{\beta\gamma}] = [L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}]$ (see Definition 2.11) defines nontrivial d–torsions $\mathbf{T}^\alpha{}_{\beta\gamma} = [L^i_{[\ jk]}, C^i{}_{ja}, \Omega^a_{ij}, T^a_{bj}, C^a{}_{[bc]}]$ and d–curvatures $\mathbf{R}^\alpha{}_{\beta\gamma\tau} = [R^i{}_{jkl}, R^a{}_{bkl}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{dbc}]$ adapted to the N–connection structure (see, respectively, the formulas (45) and (48)). It is considered that a generic off–diagonal metric $g_{\alpha\beta}$ (see Remark 2.1) is associated to a N–connection structure and reprezented as a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ (see formula (33)). The components of a N–connection and a d–metric define the canonical d–connection $\mathbf{D} = [\widehat{\mathbf{\Gamma}}^\alpha_{\beta\gamma}] = [\widehat{L}^i{}_{jk}, \widehat{L}^a{}_{bk}, \widehat{C}^i{}_{jc}, \widehat{C}^a{}_{bc}]$ (see (56)) with the corresponding values of d–torsions $\widehat{\mathbf{T}}^\alpha{}_{\beta\gamma}$ and d–curvatures $\widehat{\mathbf{R}}^\alpha{}_{\beta\gamma\tau}$. The nonmetricity d–fields are computed by using formula $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_\alpha \mathbf{g}_{\beta\gamma} = [Q_{ijk}, Q_{iab}, Q_{ajk}, Q_{abc}]$, see (35).

A.1 Generalized Lagrange–affine spaces

The Table 1 outlines seven classes of geometries modeled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure. There are emphasized the configurations:

1. Metric–affine spaces (in brief, MA) are those stated by Definition 2.9 as certain manifolds V^{n+m} of necessary smoothly class provided with arbitrary metric, $g_{\alpha\beta}$, and linear connection, $\Gamma^\alpha_{\beta\gamma}$, structures. For generic off–diagonal metrics, a MA space always admits nontrivial N–connection structures (see Proposition 3.4). Nevertheless, in general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^\alpha_{\beta\gamma}$ can be not adapted to the N–connection structure. As a consequence, the general strength fields $\left(T^\alpha{}_{\beta\gamma}, R^\alpha{}_{\beta\gamma\tau}, Q_{\alpha\beta\gamma}\right)$ can be also not N–adapted. By using the Kawaguchi's metrization process and Miron's procedure stated by Theorems 3.2 and 3.3 we can consider alternative geometries with d–connections $\mathbf{\Gamma}^\alpha_{\beta\gamma}$ (see Definition 2.11) derived from the components of N–connection and d–metric. Such geometries are adapted to the N–connection structure. They are characterized by d–torsion $\mathbf{T}^\alpha{}_{\beta\gamma}$, d–curvature $\mathbf{R}^\alpha{}_{\beta\gamma\tau}$, and nonmetricity d–field $\mathbf{Q}_{\alpha\beta\gamma}$.

2. Distinguished metric–affine spaces (DMA) are defined (see Definition 3.2) as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (33) and arbitrary d–connection $\mathbf{\Gamma}^\alpha_{\beta\gamma}$. In this case, all strenghs $\left(\mathbf{T}^\alpha{}_{\beta\gamma}, \mathbf{R}^\alpha{}_{\beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma}\right)$ are N–adapted.

3. Berwald–affine spaces (BA, see section 3.4.1) are metric–affine spaces provided with generic off–diagonal metrics with associated N–connection structure and with a Berwald d–connection ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^\alpha_{\beta\gamma}] = [\widehat{L}^i{}_{jk}, \partial_b N^a_k, 0, \widehat{C}^a{}_{bc}]$, see (62), for with the d–torsions ${}^{[B]}\mathbf{T}^\alpha{}_{\beta\gamma} = [\ {}^{[B]}L^i{}_{[jk]}, 0, \Omega^a_{ij}, T^a_{bj}, C^a{}_{[bc]}]$ and d–curvatures

$${}^{[B]}\mathbf{R}^\alpha{}_{\beta\gamma\tau} = {}^{[B]}[R^i{}_{jkl}, R^a{}_{bkl}, P^i{}_{jka}, P^c{}_{bka}, S^i{}_{jbc}, S^a{}_{dbc}]$$

are computed by introducing the components of ${}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$, respectively, in formulas (45) and (48). By definition, this space satisfies the metricity conditions on the h- and v–subspaces, $Q_{ijk}=0$ and $Q_{abc}=0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (63)).

4. Berwald–affine spaces with prescribed torsion (BAT, see sections 3.4.1 and 3.4.2) are described by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}=[L^{i}{}_{jk},\partial_b N^a_k,0,C^{a}{}_{bc}]$, with more general h– and v–components, $\widehat{L}^{i}{}_{jk}\rightarrow L^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc}\rightarrow C^{a}{}_{bc}$, inducing prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion
$$ {}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma}=[L^{i}{}_{[jk]},+\tau^{i}{}_{jk},0,\Omega^{a}_{ij},T^{a}{}_{bj},C^{a}{}_{[bc]}+\tau^{a}{}_{bc}], $$
see (65). The components of curvature ${}^{[BT]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}$ have to be computed by introducing ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ into (48). There are nontrivial components of nonmetricity d–fields, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma}=\left({}^{[B\tau]}Q_{cij},\ {}^{[B\tau]}Q_{iab}\right)$.

5. Generalized Lagrange–affine spaces (GLA, see Definition 4.6), $\mathbf{GLa}^{n}=(V^{n},g_{ij}(x,y),$ ${}^{[a]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are modeled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}=\left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ (111) are similar to those for the usual Lagrange spaces (see Definition 4.3) but with distorsions ${}^{[a]}\ \mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$. The components of d–torsions ${}^{[a]}\mathbf{T}^{\alpha}=\left(T^{i}_{jk},\tilde{T}^{i}_{jk}\right)$and d–curvatures ${}^{[a]}\mathbf{R}^{\alpha}_{.\beta\gamma\tau}=\{\ {}^{[a]}R^{i}{}_{hjk},\ {}^{[a]}P^{i}{}_{jka},\ {}^{[a]}S^{i}{}_{jbc}\}$ are computed following Theorems 4.1 and 4.2.

6. Lagrange–affine spaces (LA, see Remark 4.1), $\mathbf{La}^{n}=(V^{n},g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y)=\frac{1}{2}\frac{\partial^2L^2}{\partial y^i\partial y^j}$ (101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N}=\{\ {}^{[cL]}N^{i}_{j}\}$ (102) for a Lagrange space $\mathbf{L}^{n}=(V^{n},g_{ij}(x,y))$ (see Definition 4.2)) but with a d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha}={}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta}=\ {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta}+\ {}^{[b]}\ \mathbf{Z}^{\alpha}{}_{\beta}$. This is a particular case of GLA spaces with prescribed types of N–connection ${}^{[cL]}N^{i}_{j}$ and d–metric to be like in Lagrange geometry.

7. Finsler–affine spaces (FA, see Remark 4.2), $\mathbf{Fa}^{n}=\left(V^{n},F\left(x,y\right),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$, in their turn are introduced by further restrictions of $\mathbf{La}^{n}$ to a quadratic form $g^{[F]}_{ij}=\frac{1}{2}\frac{\partial^2F^2}{\partial y^i\partial y^j}$ (85) constructed from a Finsler metric $F\left(x^{i},y^{j}\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N}=\{\ {}^{[F]}N^{i}_{j}\}$ (86) as in the Finsler space $\mathbf{F}^{n}=(V^{n},F\left(x,y\right))$ but with a d–connection structure ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}=\ {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta}+\ {}^{[f]}\ \mathbf{Z}^{\alpha}{}_{\beta}$,where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection (88).

A.2 Generalized Hamilton–affine spaces

The Table 2 outlines geometries modeled in the framework of metric–affine geometry as spaces with nontrivial N–connection structure splitting the space into any conventional a

horizontal subspace and vertical subspace being isomorphic to a dual vector space provided with respective dual coordinates. We can use respectively the classification from Table 1 when the v–subspace is transformed into dual one as we noted in Remark 4.3 For simplicity, we label such spaces with symbols like $\check{N}_{ai}$ instead N_i^a where "inverse hat" points that the geometric object is defined for a space containing a dual subspaces. The local h–coordinates are labeled in the usual form, x^i, with $i = 1, 2, ..., n$ but the v–coordinates are certain dual vectors $\check{y}^a = p_a$ with $a = n+1, n+2, ..., n+m$. The local coordinates are denoted $\check{u}^\alpha = (x^i, \check{y}^a) = (x^i, p_a)$. The curvature of a N–connection $\check{N}_{ai}$ is computed as $\check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$. The h– v–irreducible components of a general d–connection are parametrized $\check{\mathbf{D}} = [\check{\mathbf{\Gamma}}^\alpha_{\beta\gamma}] = [L^i{}_{jk}, L_a{}^b{}_k, \check{C}^i{}_j{}^c, \check{C}_a{}^{bc}]$, the d–torsions are $\check{\mathbf{T}}^\alpha{}_{\beta\gamma} = [L^i{}_{[jk]}, L_a{}^b{}_k, \check{C}^i{}_j{}^c, \check{C}_a{}^{[bc]}]$ and the d–curvatures ${}^{[B]}\check{\mathbf{R}}^\alpha{}_{\beta\gamma\tau} = [R^i{}_{jkl}, \check{R}_a{}^b{}_{kl}, \check{P}^i{}_{jk}{}^a, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a{}^{dbc}]$. The nonmetricity d–fields are stated $\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\check{\mathbf{D}}_\alpha \check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk}, \check{Q}_i{}^{ab}, \check{Q}^a{}_{jk}, \check{Q}^{abc}]$. There are also considered additional labels for the Berwald, Cartan and another type d–connections.

1. Metric–dual–affine spaces (in brief, MDA) are usual metric–affine spaces with a prescribed structure of "dual" local cooridnates.

2. Distinguished metric–dual-affine spaces (DMDA) are provided with d–metric and d–connection structures adapted to a N–connection $\check{N}_{ai}$ defining a global splitting into a usual h–subspace and a v–dual–subspace being dual to a usual v–subspace.

3. Berwald–dual–affine spaces (BDA) are Berwald–affine spaces with a dual v–subspace. Their Berwald d–connection is stated in the form
$$ {}^{[B]}\check{\mathbf{D}} = [{}^{[B]}\check{\mathbf{\Gamma}}^\alpha_{\beta\gamma}] = [\widehat{L}^i{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_a{}^{[bc]}] $$
with induced d–torsions ${}^{[B]}\check{\mathbf{T}}^\alpha{}_{\beta\gamma} = [L^i_{[jk]}, 0, \check{\Omega}_{iaj}, \check{T}_a{}^b{}_j, \check{C}_a{}^{[bc]}]$ and d–curvatures
$$ {}^{[B]}\check{\mathbf{R}}^\alpha{}_{\beta\gamma\tau} = [R^i{}_{jkl}, \check{R}_a{}^b{}_{kl}, , \check{P}^i{}_{jk}{}^a, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a{}^{dbc}] $$
computed by introducing the components of ${}^{[B]}\check{\mathbf{\Gamma}}^\alpha_{\beta\gamma}$, respectively, in formulas (45) and (48) re–defined for dual v–subspaces. By definition, this d–connection satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $\check{Q}^{abc} = 0$ but with nontrivial components of ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\,{}^{[B]}\check{\mathbf{D}}_\alpha \check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk} = 0, \check{Q}_i{}^{ab}, \check{Q}^a{}_{jk}, \check{Q}^{abc} = 0]$.

4. Berwald–dual–affine spaces with prescribed torsion (BDAT) are described by a more general class of d–connections ${}^{[BT]}\check{\mathbf{\Gamma}}^\alpha_{\beta\gamma} = [L^i{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_a{}^{bc}]$, inducing prescribed values $\tau^i{}_{jk}$ and $\check{\tau}_a{}^{bc}$ for d–torsions
$$ {}^{[BT]}\check{\mathbf{T}}^\alpha{}_{\beta\gamma} = [L^i{}_{[jk]} + \tau^i{}_{jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_a{}^b{}_j, \check{C}_a{}^{[bc]} + \check{\tau}_a{}^{bc}]. $$
The components of d–curvatures
$$ {}^{[BT]}\check{\mathbf{R}}^\alpha{}_{\beta\gamma\tau} = \ [R^i{}_{jkl}, \check{R}_a{}^b{}_{kl}, \check{P}^i{}_{jk}{}^a, \check{P}_c{}^b{}_k{}^a, \check{S}^i{}_j{}^{bc}, \check{S}_a{}^{dbc}] $$
have to be computed by introducing ${}^{[BT]}\check{\mathbf{\Gamma}}^\alpha_{\beta\gamma}$ into dual form of formulas (48). There are nontrivial components of nonmetricity d–field, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = -\,{}^{[BT]}\check{\mathbf{D}}_\alpha \check{\mathbf{g}}_{\beta\gamma} = (Q_{ijk} = 0, \check{Q}_i{}^{ab}, \check{Q}^a{}_{jk}, \check{Q}^{abc} = 0)$.

5. Generalized Hamilton–affine spaces (GHA), $\mathbf{GHa}^n = \left(V^n, \check{g}^{ij}(x,p), \ ^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, are modeled as distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a cotangent bundle structure. The d–metric $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ab}]$ and the d–connection $\ ^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\ ^{[a]}L^i_{jk}, \ ^{[a]}\check{C}_i{}^{jc})$ are similar to those for usual Hamilton spaces (see section 4.3.2) but with distorsions $\ ^{[a]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields $\ ^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The components of d–torsion and d–curvature, respectively, $\ ^{[a]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^i{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_a{}^{[bc]}]$ and $\ ^{[a]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = [R^i{}_{jkl}, \check{P}^i{}_{jk}^{\ a}, \check{S}_a{}^{dbc}]$, are computed following Theorems 4.1 and 4.2 reformulated for cotangent bundle structures.

6. Hamilton–affine spaces (HA, see Remark 4.3), $\mathbf{Ha}^n = (V^n, \check{g}^{ij}_{[H]}(x,p), \ ^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta})$, are provided with Hamilton N–connection $\ ^{[H]}\check{N}_{ij}(x,p)$ (110) and quadratic form $\check{g}^{ij}_{[H]}$ (109) for a Hamilton space $\mathbf{H}^n = (V^n, H(x,p))$ (see section 4.3.2)) but with a d–connection structure $\ ^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \ ^{[H]}[L^i{}_{jk}, \check{C}_a{}^{bc}]$ distorted by arbitrary torsion, $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$, and nonmetricity d–fields, $\check{\mathbf{Q}}_{\beta\gamma\alpha}$, when $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta} = \ ^{[H]}\widehat{\check{\mathbf{\Gamma}}}^{\alpha}{}_{\beta} + \ ^{[H]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$. This is a particular case of GHA spaces with prescribed types of N–connection $\ ^{[H]}\check{N}_{ij}$ and d–metric $\check{\mathbf{g}}^{[H]}_{\alpha\beta} = [g^{ij}_{[H]} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_i}]$ to be like in the Hamilton geometry.

7. Cartan–affine spaces (CA, see Remark 4.3), $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p), \ ^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, are dual to the Finsler spaces $\mathbf{Fa}^n = \left(V^n, F(x,y), \ ^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$. The CA spaces are introduced by further restrictions of $\mathbf{Ha}^n$ to a quadratic form $\check{g}^{ij}_{[C]}$ (98) and canonical N–connection $\check{N}^{[C]}_{ij}$ (99). They are like usual Cartan spaces, see section 4.2.5) but contain distorsions induced by nonmetricity $\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The d–metric is parametrized $\check{\mathbf{g}}^{[C]}_{\alpha\beta} = [g^{ij}_{[C]} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_i}]$ and the curvature $\ ^{[C]}\check{\Omega}_{iaj}$ of N–connection $\ ^{[C]}\check{N}_{ia}$ is computed $\ ^{[C]}\check{\Omega}_{iaj} = \delta_{[i}\ ^{[C]}\check{N}_{j]a}$. The Cartan's d–connection $\ ^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \ ^{[C]}[L^i{}_{jk}, L^i{}_{jk}, \check{C}_a{}^{bc}, \check{C}_a{}^{bc}]$ possess nontrivial d–torsions $\ ^{[C]}\check{\mathbf{T}}_{\alpha}{}^{\beta\gamma} = [L^i{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_a{}^{[bc]}]$ and d–curvatures $\ ^{[C]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = [R^i{}_{jkl}, \check{P}^i{}_{jk}^{\ a}, \check{S}_a{}^{dbc}]$ computed following Theorems 4.1 and 4.2 reformulated on cotangent bundles with explicit type of N–connection $\check{N}^{[C]}_{ij}$ d–metric $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$ and d–connection $\ ^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. The nonmetricity d–fields are not trivial for such spaces, $\ ^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ ^{[C]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk}, \check{Q}_i{}^{ab}, \check{Q}^a{}_{jk}, \check{Q}^{abc}]$.

A.3 Teleparallel Lagrange–affine spaces

We considered the main properties of teleparallel Finsler–affine spaces in section 4.2.4 (see also section 4.1.3 on locally isotropic teleparallel spaces). Every type of teleparallel spaces is distinguished by the condition that the curvature tensor vanishes but the torsion plays a cornstone role. Modeling generalized Finsler structures on metric–affine spaces, we do not impose the condition on vanishing nonmetricity (which is stated for usual teleparallel spaces). For $\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$, the classification of spaces from Table 1 trasforms in that from Table 3.

1. Teleparallel metric–affine spaces (in brief, TMA) are usual metric–affine ones but with vanishing curvature, modeled on manifolds V^{n+m} of necessary smoothly class

provided, for instance, with the Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ (82). For generic off–diagonal metrics, a TMA space always admits nontrivial N–connection structures (see Proposition 3.4). We can model teleparallel geometries with local anisotropy by distorting the Levi–Civita or the canonical d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (see Definition 2.11) both constructed from the components of N–connection and d–metric. In general, such geometries are characterized by d–torsion $\mathbf{T}^{\alpha}{}_{\beta\gamma}$ and nonmetricity d–field $\mathbf{Q}_{\alpha\beta\gamma}$ both constrained to the condition to result in zero d–curvatures.

2. Distinguished teleparallel metric–affine spaces (DTMA) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (33) and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with vanishing d–curvatures defined by Weitzenbock–affine d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions by nonmetricity d–fields preserving the condition of zero values for d–curvatures.

3. Teleparallel Berwald–affine spaces (TBA) are defined by distorsions of the Weitzenbock connection to any Berwald like strucutre, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. All constructions with generic off–diagonal metrics can be adapted to the N–connection and considered for d–objects. By definition, such spaces satisfy the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (63)).

4. Teleparallel Berwald–affine spaces with prescribed torsion (TBAT) are defined by a more general class of distorsions resulting in the Weitzenbock type d–connections, ${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$, with more general h– and v–components, $\widehat{L}^{i}{}_{jk} \to L^{i}{}_{jk}$ and $\widehat{C}^{a}{}_{bc} \to C^{a}{}_{bc}$, having prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion ${}^{[WB]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}]$ and characterized by the condition ${}^{[WB\tau]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$ with notrivial components of nonmetricity ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij},\ Q_{iab})$.

5. Teleparallel generalized Lagrange–affine spaces (TGLA) are distinguished metric–affine spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure with zero d–curvature. The Weitzenblock–Lagrange d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left({}^{[Wa]}L^{i}_{jk},\ {}^{[Wa]}C^{i}_{jc}\right)$, where ${}^{[WaL]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ is defined by a d–metric $\mathbf{g}_{[a]}$ (112) $\mathbf{Z}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$ and ${}^{[Wa]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$.

6. Teleparallel Lagrange–affine spaces (TLA 4.1) consist a subclass of spaces $\mathbf{La}^{n} = \left(V^{n}, g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}{}_{\beta}\right)$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^{i}_{j}\}$ (102) for a Lagrange space $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, ${}^{[W]}\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$. This is a generalization of teleparallel Finsler affine spaces (see section (4.2.4)) when $g^{[L]}_{ij}(x,y)$ is considered instead of $g^{[F]}_{ij}(x,y)$.

7. Teleparallel Finsler–affine spaces (TFA) are particular cases of spaces of type $\mathbf{Fa}^n = (V^n, F(x,y), {}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta})$, defined by a quadratic form $g_{ij}^{[F]} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) constructed from a Finsler metric $F(x^i, y^j)$. They are provided with a canonical N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N^i_j\}$ (86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ but with a Finsler–Weitzenbock d–connection structure ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$, respective d–torsion, ${}^{[WF]}\mathbf{T}_{\beta}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown}{}_{\beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$,where $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ contains distorsions from the canonical Finsler d–connection (88). Such distorsions are constrained to satisfy the condition of vanishing curvature d–tensors (see section (4.2.4)).

A.4 Teleparallel Hamilton–affine spaces

This class of metric–affine spaces is similar to that outlined in previous subsection, see Table 3 but derived on spaces with dual vector bundle structure and induced generalized Hamilton–Cartan geometry (section 4.3.2 and Remark 4.3). We outline the main denotations for such spaces and note that they are characterized by the condition $\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau} = 0$.

1. Teleparallel metric dual affine spaces (in brief, TMDA) define teleparalles structures on metric–affine spaces provided with generic off–diagonal metrics and associated N–connections modeling splittings with effective dual vector bundle structures.

2. Distinguished teleparallel metric dual affine spaces (DTMDA) are spaces provided with independent d–metric, d–connection structures adapted to a N–connection in an effective dual vector bundle and resulting in zero d–curvatures.

3. Teleparallel Berwald dual affine spaces (TBDA) .

4. Teleparallel dual Berwald–affine spaces with prescribed torsion (TDBAT).

5. Teleparallel dual generalized Hamilton–affine spaces (TDGHA).

6. Teleparallel dual Hamilton–affine spaces (TDHA, see section 4.1).

7. Teleparallel dual Cartan–affine spaces (TDCA).

A.5 Generalized Finsler–Lagrange spaces

This class of geometries is modeled on vector/tangent bundles [14] (see subsections 4.2 and 4.3.1) or on metric–affine spaces provided with N–connection structure. There are also alternative variants when metric–affine structures are defined for vector/tangent bundles with independent generic off–diagonal metrics and linear connection structures. The standard approaches to generalized Finsler geometries emphasize the connections satisfying the metricity conditions. Nevertherless, the Berwald type connections admit certain nonmetricity d–fields. The classification stated in Table 5 is similar to that from Table 1 with that difference that the spaces are defined from the very beginning to be any vector or tangent bundles. The local coordinates x^i are considered for base subspaces and y^a are for fiber type subspaces. We list the short denotations and main properties of such spaces:

1. Metric affine vector bundles (in brief, MAVB) are provieded with arbitrary metric $g_{\alpha\beta}$ and linear connection $\Gamma^{\alpha}_{\beta\gamma}$ structure. For generic off–diagonal metrics, we can introduce associated nontrivial N–connection structures. In general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$, but $\Gamma^{\alpha}_{\beta\gamma}$ may be not adapted to the N–connection structure. As a consequence, the general strength fields $\left(T^{\alpha}_{\ \beta\gamma}, R^{\alpha}_{\ \beta\gamma\tau}, Q_{\alpha\beta\gamma} = 0\right)$, defined in the total space of the vector bundle are also not N–adapted. We can consider a metric–affine (MA) structure on the total space if $Q_{\alpha\beta\gamma} \neq 0$.

2. Distinguished metric–affine vector bundles (DMAVB) are provided with N–connection structure N^a_i, d–metric field and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$. In this case, all strengths $(\mathbf{T}^{\alpha}_{\ \beta\gamma}, \mathbf{R}^{\alpha}_{\ \beta\gamma\tau}, \mathbf{Q}_{\alpha\beta\gamma} = 0)$ are N–adapted. A distinguished metric–affine (DMA) structure on the total space is considered if $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$.

3. Berwald metric–affine tangent bundles (BMATB) are provided with Berwald d–connection structure ${}^{[B]}\mathbf{\Gamma}$. By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} do not vanish (see formulas (63)).

4. Berwald metric–affine bundles with prescribed torsion (BMATBT) are described by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^i_{\ jk}, \partial_b N^a_k, 0, C^a_{\ bc}]$ inducing prescribed values $\tau^i_{\ jk}$ and $\tau^a_{\ bc}$ in d–torsion

$$
{}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = [L^i_{\ [jk]}, +\tau^i_{\ jk}, 0, \Omega^a_{ij}, T^a_{\ bj}, C^a_{\ [bc]} + \tau^a_{\ bc}],
$$

see (65). There are nontrivial nonmetricity d–fields, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij}, Q_{iab})$.

5. Generalized Lagrange metric–affine bundles (GLMAB) are modeled as $\mathbf{GLa}^n = (V^n,$ $g_{ij}(x,y),\ {}^{[a]}\mathbf{\Gamma}^{\alpha}_{\ \beta})$ spaces on tangent bundles provided with generic off–diagonal metrics with associated N–connection. If the d–connection is a canonical one, $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$, the nonmetricity vanish. But we can consider arbitrary d–connections $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with nontrivial nonmetricity d–fields.

6. Lagrange metric–affine bundles (LMAB) are defined on tangent bundles as spaces $\mathbf{La}^n = \left(V^n, g^{[L]}_{ij}(x,y),\ {}^{[b]}\mathbf{\Gamma}^{\alpha}_{\ \beta}\right)$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{\ {}^{[cL]}N^i_j\}$ for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ (see Definition 4.2)) but with a d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}\vartheta^{\beta}$ distorted by arbitrary torsion, $\mathbf{T}_{\beta}$, and nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$, when ${}^{[b]}\mathbf{\Gamma}^{\alpha}_{\ \beta} = {}^{[L]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta} + {}^{[b]}\mathbf{Z}^{\alpha}_{\ \ \beta}$. This is a particular case of GLA spaces with prescribed types of N–connection ${}^{[cL]}N^i_j$ and d–metric to be like in Lagrange geometry.

7. Finsler metric–affine bundles (FMAB), are modeled on tangent bundles as spaces $\mathbf{Fa}^n = \left(V^n, F(x,y),\ {}^{[f]}\mathbf{\Gamma}^{\alpha}_{\ \beta}\right)$ with quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) constructed from a Finsler metric $F(x^i, y^j)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \{\ {}^{[F]}N^i_j\}$ as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ but with a d–connection

structure ${}^{[f]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ distorted by arbitrary torsion, $\mathbf{T}^{\alpha}_{\beta\gamma}$, and nonmetricity, $\mathbf{Q}_{\beta\gamma\tau}$, d–fields, ${}^{[f]}\mathbf{\Gamma}^{\alpha}{}_{\beta} = {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta} + {}^{[f]}\mathbf{Z}^{\alpha}{}_{\beta}$,where ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ is the canonical Finsler d–connection (88).

A.6 Generalized Hamilton–Cartan spaces

Such spaces are modeled on vector/tangent dual bundles (see sections subsections 4.3.2 and 4.2.5) as metric–affine spaces provided with N–connection structure. The classification stated in Table 6 is similar to that from Table 2 with that difference that the geometry is modeled from the very beginning as vector or tangent dual bundles. The local coordinates x^i are considered for base subspaces and $y^a = p_a$ are for cofiber type subspaces. So, the spaces from Table 6 are dual to those from Table 7, when the respective Lagrange–Finsler structures are changed into Hamilton–Cartan structures. We list the short denotations and main properties of such spaces:

1. The metric–affine dual vector bundles (in brief, MADVB) are defined by metric–affine independent metric and linear connection structures stated on dual vector bundles. For generic off–diagonal metrics, there are nontrivial N–connection structures. The linear connection may be not adapted to the N–connection structure.

2. Distinguished metric-affine dual vector bundles (DMADVB) are provided with d–metric and d–connection structures adapted to a N–connection $\check{N}_{ai}$.

3. Berwald metric–affine dual bundles (BMADB) are provided with a Berwald d–connection
$$ {}^{[B]}\check{\mathbf{D}} = [\ {}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_{a}{}^{[bc]}]. $$
By definition, on such spaces, there are satisfied the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $\check{Q}^{abc} = 0$ but with nontrivial components of ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[B]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = [Q_{ijk} = 0, \check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}, \check{Q}^{abc} = 0]$.

4. Berwald metricl–affine dual bundles with prescribed torsion (BMADBT) are described by a more general class of d–connections ${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_{a}{}^{bc}]$ inducing prescribed values $\tau^{i}{}_{jk}$ and $\check{\tau}_{a}{}^{bc}$ for d–torsions
$$ {}^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]} + \tau^{i}{}_{jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_{a}{}^{b}{}_{j}, \check{C}_{a}{}^{[bc]} + \check{\tau}_{a}{}^{bc}]. $$
There are nontrivial components of nonmetricity d–field, ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = {}^{[BT]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}_{\beta\gamma} = \left(Q_{ijk} = 0, \check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}, \check{Q}^{abc} = 0\right)$.

5. Generalized metric–affine Hamilton bundles (GMAHB) are modeled on dual vector bundles as spaces $\mathbf{GHa}^{n} = \left(V^{n}, \check{g}^{ij}(x,p), {}^{[a]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, provided with generic off–diagonal metrics with associated N–connection inducing a cotangent bundle structure. The d–metric $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ab}]$ and the d–connection ${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left({}^{[a]}L^{i}_{jk}, {}^{[a]}\check{C}_{i}{}^{jc}\right)$ are similar to those for usual Hamilton spaces, with distorsions ${}^{[a]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta}$ inducing general nontrivial nonmetricity d–fields ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$. For canonical configurations, ${}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$, we obtain ${}^{[GH]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$.

6. Metric–affine Hamilton bundles (MAHB) are defied on dual bundles as spaces $\mathbf{Ha}^n = \left(V^n, \check{g}^{ij}_{[H]}(x,p),\ {}^{[b]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$, provided with Hamilton N–connection ${}^{[H]}\check{N}_{ij}(x,p)$ and quadratic form $\check{g}^{ij}_{[H]}$ for a Hamilton space $\mathbf{H}^n = (V^n, H(x,p))$ (see section 4.3.2) with a d–connection structure ${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[H]}[L^{i}{}_{jk}, \check{C}_{a}{}^{bc}]$ distorted by arbitrary torsion, $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$, and nonmetricity d–fields, $\check{\mathbf{Q}}_{\beta\gamma\alpha}$, when $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta} = {}^{[H]}\widehat{\check{\mathbf{\Gamma}}}^{\alpha}{}_{\beta} + {}^{[H]}\check{\mathbf{Z}}^{\alpha}{}_{\beta}$. This is a particular case of GMAHB spaces with prescribed types of N–connection ${}^{[H]}\check{N}_{ij}$ and d–metric $\check{\mathbf{g}}^{[H]}_{\alpha\beta} = [g^{ij}_{[H]} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_i}]$ to be like in the Hamilton geometry but with nontrivial nonmetricity.

7. Metric–affine Cartan bundles (MACB) are modeled on dual tangent bundles as spaces $\mathbf{Ca}^n = \left(V^n, \check{g}^{ij}_{[K]}(x,p),\ {}^{[c]}\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta}\right)$ being dual to the Finsler spaces. They are like usual Cartan spaces, see section 4.2.5) but may contain distorsions induced by nonmetricity $\check{\mathbf{Q}}_{\alpha\beta\gamma}$. The d–metric is parametrized $\check{\mathbf{g}}^{[C]}_{\alpha\beta} = [g^{ij}_{[C]} = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i \partial p_i}]$ and the curvature ${}^{[C]}\check{\Omega}_{iaj}$ of N–connection ${}^{[C]}\check{N}_{ia}$ is computed ${}^{[C]}\check{\Omega}_{iaj} = \delta_{[i}\ {}^{[C]}\check{N}_{j]a}$. The Cartan's d–connection ${}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = {}^{[C]}[L^{i}{}_{jk}, L^{i}{}_{jk}, \check{C}_{a}{}^{bc}, \check{C}_{a}{}^{bc}]$ possess nontrivial d–torsions ${}^{[C]}\check{\mathbf{T}}_{\alpha}{}^{\beta\gamma} = [L^{i}{}_{[jk]}, \check{\Omega}_{iaj}, \check{C}_{a}{}^{[bc]}]$ and d–curvatures ${}^{[C]}\check{\mathbf{R}}^{\alpha}_{.\beta\gamma\tau} = {}^{[C]}[R^{i}{}_{jkl}, \check{P}^{i}{}_{jk}{}^{a}, \check{S}_{a}{}^{dbc}]$ computed following Theorems 4.1 and 4.2 reformulated on cotangent bundles with explicit type of N–connection $\check{N}^{[C]}_{ij}$ d–metric $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$ and d–connection ${}^{[C]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. Distorsions result in d–connection $\check{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma} = {}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} + {}^{[C]}\ \check{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$. The nonmetricity d–fields are not trivial for such spaces.

A.7 Teleparallel Finsler–Lagrange spaces

The teleparallel configurations can be modeled on vector and tangent bundles (the teleparallel Finsler–affine spaces are defined in section 4.2.4, see also section 4.1.3 on locally isotropic teleparallel spaces) were constructed as subclasses of metric–affine spaces on manifolds of necessary smoothly class. The classification from Table 7 is a similar to that from Table 3 but for direct vector/ tangent bundle configurations with vanishing nonmetricity. Nevertheless, certain nonzero nonmetricity d–fields can be present if the Berwald d–connection is considered or if we consider a metric–affine geometry in bundle spaces.

1. Teleparallel vector bundles (in brief, TVB) are provided with independent metric and linear connection structures like in metric–affine spaces satisfying the condition of vanishing curvature. The N–connection is associated to generic off–diagonal metrics. The TVB spaces can be provided with a Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$(82) which can be transformed in a d–connection one with respect to N–adapted frames. We can model teleparallel geometries with local anisotropy by distorting the Levi–Civita or the canonical d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (see Definition 2.11) both constructed from the components of N–connection and d–metric. In general, such vector (in particular cases, tangent) bundle geometries are characterized by d–torsions $\mathbf{T}^{\alpha}{}_{\beta\gamma}$ and nonmetricity d–fields $\mathbf{Q}_{\alpha\beta\gamma}$ both constrained to the condition to result in zero d–curvatures.

2. Distinguished teleparallel vector bundles (DTVB, or vect. b.) are provided with N–connection structure N^{a}_{i}, d–metric field (33) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with

vanishing d–curvatures. The geometric constructions are stated by the Weitzenbock–affine d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions by nonmetricity d–fields preserving the condition of zero values for d–curvatures. The standard constructions from Finsler geometry and generalizations are with vanishing nonmetricity.

3. Teleparallel Berwald vector bundles (TBVB) are defined by Weitzenbock connections of Berwald type strucutre, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. By definition, such spaces satisfy the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} do not vanish (see formulas (63)).

4. Teleparallel Berwald vector bundles with prescribed torsion (TBVBT) are defined by a more general class of distorsions alsow resulting in the Weitzenbock d–connection, ${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion,

$$ {}^{[WB]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}], $$

characterized by the condition ${}^{[WB\tau]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$ and nontrivial components of nonmetricity d–field, ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = (Q_{cij}, Q_{iab})$.

5. Teleparallel generalized Lagrange spaces (TGL) are modeled on tangent bundles (tang. b.) provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure being enabled with zero d–curvature. The Weitzenblock–Lagrange d–connections ${}^{[Wa]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left({}^{[Wa]}L^{i}_{jk},\ {}^{[Wa]}C^{i}_{jc} \right),\quad {}^{[WaL]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ are defined by a d–metric $\mathbf{g}_{[a]}$ (112) $\mathbf{Z}^{\alpha}{}_{\beta}$ inducing ${}^{[Wa]}\mathbf{R}^{\alpha}{}_{\beta\gamma\tau} = 0$. For simplicity, we consider the configurations when nonmetricity d–fields ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$.

6. Teleparallel Lagrange spaces (TL) are modeled on tangent bundles provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{ {}^{[cL]}N^{i}_{j}\}$ (102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, ${}^{[W]}\mathbf{T}_{\beta}$ when ${}^{[WL]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$. For simplicity, we can consider configurations with zero nonmetricity d–fields, $\mathbf{Q}_{\beta\gamma\alpha}$.

7. Teleparallel Finsler spaces (TF) are modeled on tangent bundles provided with a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) constructed from a Finsler metric $F(x^i, y^j)$. They are also enabled with a canonical N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N^{i}_{j}\}$ (86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ but with a Finsler–Weitzenbock d–connection structure ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}$, respective d–torsion, ${}^{[WF]}\mathbf{T}_{\beta}$. We can write ${}^{[WF]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$,where $\hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ contains distorsions from the canonical Finsler d–connection (88). Such distorsions are constrained to satisfy the condition of vanishing curvature d–tensors (see section (4.2.4)) and, for simplicity, of vanishing nonmetricity, $\mathbf{Q}_{\beta\gamma\tau} = 0$.

A.8 Teleparallel Hamilton–Cartan spaces

This subclass of Hamilton–Cartan spaces is modeled on dual vector/ tangent bundles being similar to that outlined in Table 4 (on generalized Hamilton–Cartan geometry, see section 4.3.2 and Remark 4.3) and dual to the subclass outlined in Table 7. We outline the main denotations and properties of such spaces and note that they are characterized by the condition $\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau} = 0$ and $\check{\mathbf{Q}}^{\alpha}{}_{\beta\gamma} = 0$ with that exception that there are nontrivial nonmetricity d–fields for Berwald configuratons.

1. Teleparallel dual vector bundles (TDVB, or d. vect. b.) are provided with generic off–diagonal metrics and associated N–connections. In general, $\check{Q}^{\alpha}{}_{\beta\gamma} \neq 0$.

2. Distinguished teleparallel dual vector bundles spaces (DTDVB) are provided with independent d–metric, d–connection structures adapted to a N–connection in an effective dual vector bundle and resulting in zero d–curvatures. In general, $\check{\mathbf{Q}}^{\alpha}{}_{\beta\gamma} \neq 0$.

3. Teleparallel Berwald dual vector bundles (TBDVB) are provided with Berwald–Weitzenbock d–connection structure resulting in vanishing d–curvature.

4. Teleparallel Berwald dual vector bundles with prescribed d–torsion (TBDVB) are with d–connections ${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b \check{N}_{ai}, 0, \check{C}_{a}{}^{bc}]$ inducing prescribed values $\tau^{i}{}_{jk}$ and $\check{\tau}_{a}{}^{bc}$ for d–torsions ${}^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]} + \tau^{i}{}_{jk}, 0, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}, T_{a}{}^{b}{}_{j}, \check{C}_{a}{}^{[bc]} + \check{\tau}_{a}{}^{bc}]$. They are described by certain distorsions to a Weitzenbock d–connection.

5. Teleparallel generalized Hamilton spaces (TGH) consist a subclass of generalized Hamilton spaces with vanishing d–curvature structure, defined on dual tangent bundles (d. tan. b.). They are described by distorsions to a Weitzenbock d–connection ${}^{[Wa]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$. In the simplest case, we consider ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$.

6. Teleparallel Hamilton spaces (TH, see section 4.1), as a particular subclass of TGH, are provided with d–connection and N–connection structures corresponding to Hamilton configurations.

7. Teleparallel Cartan spaces (TC) are particular Cartan configurations with absolut teleparallelism.

A.9 Distinguished Riemann–Cartan spaces

A wide class of generalized Finsler geometries can be modeled on Riemann–Cartan spaces by using generic off–diagonal metrics and associated N–connection structures. The locally anisotropic metric–affine configurations from Table 1 transform into a Riemann–Cartan ones if we impose the condition of metricity. For the Berwald type connections one could be certain nontrivial nonmetricity d–fields on intersection of h- and v–subspaces. The local coordinates x^i are considered as certain holonomic ones and y^a are anholonomic. We list the short denotations and main properties of such spaces:

1. Riemann–Cartan spaces (in brief, RC, see related details in section 3.5.1) are certain manifolds V^{n+m} of necessary smoothly class provided with metric structure $g_{\alpha\beta}$ and linear connection structure $\Gamma^{\alpha}_{\beta\gamma}$ (constructed as a distorsion by torsion of the Levi–Civita connection) both satisfying the conditions of metric compatibility, $Q_{\alpha\beta\gamma}=0$. For generic off–diagonal metrics, a RC space always admits nontrivial N–connection structures (see Proposition 3.4 reformulated for the case of vanishing nonmetricity). In general, only the metric field $g_{\alpha\beta}$ can be transformed into a d–metric one, $\mathbf{g}_{\alpha\beta}=[g_{ij},h_{ab}]$, but $\Gamma^{\alpha}_{\beta\gamma}$ may be not adapted to the N–connection structure.

2. Distinguished Riemann–Cartan spaces (DRC) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (33) and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (a distorsion of the Levi–Civita connection, or of the canonical d–connection) satisfying the condition $\mathbf{Q}_{\alpha\beta\gamma}=0$. In this case, the strengths $\left(\mathbf{T}^{\alpha}{}_{\beta\gamma},\mathbf{R}^{\alpha}{}_{\beta\gamma\tau}\right)$ are N–adapted.

3. Berwald Riemann–Cartan (BRC) are modeled if a N–connection structure is defined in a Riemann–Cartan space and distorting the connection to a Berwald d–connection ${}^{[B]}\mathbf{D}=[\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}]=[\widehat{L}^i{}_{jk},\partial_b N^a_k,0,\widehat{C}^a{}_{bc}]$, see (62). By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk}=0$ and $Q_{abc}=0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (63)). Nonmetricities vanish with respect to holonomic frames.

4. Berwald Riemann–Cartan spaces with prescribed torsion (BRCT) are defined by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}=[L^i{}_{jk},\partial_b N^a_k,0,C^a{}_{bc}]$ inducing prescribed values $\tau^i{}_{jk}$ and $\tau^a{}_{bc}$ in d–torsion ${}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma}=[L^i{}_{[jk]},+\tau^i{}_{jk},0,\Omega^a_{ij},T^a{}_{bj},C^a{}_{[bc]}+\tau^a{}_{bc}]$, see (65). The nontrivial components of nonmetricity d–fields are ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma}=(Q_{cij},Q_{iab})$. Such components vanish with respect to holonomic frames.

5. Generalized Lagrange Riemann–Cartan spaces (GLRC) are modeled as distinguished Riemann–Cartan spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}=\left(\ {}^{[a]}L^i_{jk},\ {}^{[a]}C^i_{jc}\right)$ (111) are those for the usual Lagrange spaces (see Definition 4.3).

6. Lagrange Riemann–Cartan spaces (LRC, see Remark 4.1) are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y)=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N}=\{\ {}^{[cL]}N^i_j\}$ (102) for a Lagrange space $\mathbf{L}^n=(V^n,g_{ij}(x,y))$ (see Definition 4.2)) and, for instance, with a canonical d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha}={}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ satisfying metricity conditions for the d–metric defined by $g^{[L]}_{ij}(x,y)$.

7. Finsler Riemann–Cartan spaces (FRC, see Remark 4.2) are defined by a quadratic form $g^{[F]}_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) constructed from a Finsler metric $F\left(x^i,y^j\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N}=\{\ {}^{[F]}N^i_j\}$ (86) as in the Finsler space $\mathbf{F}^n=(V^n,F(x,y))$ with ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ being the canonical Finsler d–connection (88).

A.10 Distinguished (pseudo) Riemannian spaces

Sections 3.5.1 and 4.1.1 are devoted to modeling of locally anisotropic geometric configurations in (pseudo) Riemannian spaces enabled with generic off–diagonal metrics and associated N–connection structure. Different classes of generalized Finsler metrics can be embedded in (pseudo) Riemannian spaces as certain anholonomic frame configurations. Every such space is characterized by a corresponding off–diagonal metric ansatz and Levi–Civita connection stated with respect to coordinate frames or, alternatively (see Theorem 3.4), by certain N–connection and induced d–metric and d–connection structures related to the Levi–Cevita connection with coefficients defined with respect to N–adapted anholonomic (co) frames. We characterize every such type of (pseudo) Riemannian spaces both by Levi–Civita and induced canonical/or Berwald d–connections which contain also induced (by former off–diagonal metric terms) nontrivial d–torsion and/or nonmetricity d–fields.

1. (Pseudo) Riemann spaces (in brief, pR) are certain manifolds V^{n+m} of necessary smoothly class provided with generic off–diagonal metric structure $g_{\alpha\beta}$ of arbitrary signature inducing the unique torsionless and metric Levi–Civita connection $\Gamma^{\alpha}_{\bigtriangledown\beta\gamma}$. We can effectively diagonalize such metrics by anholonomic frame transforms with associated N–connection structure. We can also consider alternatively the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, L^{a}{}_{bk}, C^{i}{}_{jc}, C^{a}{}_{bc}]$ (56)defined by the coefficients of d–metric $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ and N–connection N^{a}_{i}. We have nontrivial d–torsions $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$, but $T^{\alpha}_{\bigtriangledown\beta\gamma} = 0, Q^{\bigtriangledown}_{\alpha\beta\gamma} = 0$ and $\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$. The simplest anholonomic configurations are characterized by associated N–connections with vanishing N–connection curvature, $\Omega^{a}_{ij} = \delta_{[i}N^{a}_{j]} = 0$. The d–torsions $\widehat{\mathbf{T}}^{\alpha}{}_{\beta\gamma} = [\widehat{L}^{i}_{[\ jk]}, \widehat{C}^{i}{}_{ja}, \Omega^{a}_{ij}, \widehat{T}^{a}{}_{bj}, \widehat{C}^{a}{}_{[bc]}]$ and d–curvatures $\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau} = [\widehat{R}^{i}{}_{jkl}, \widehat{R}^{a}{}_{bkl}, \widehat{P}^{i}{}_{jka}, \widehat{P}^{c}{}_{bka}, \widehat{S}^{i}{}_{jbc}, \widehat{S}^{a}{}_{dbc}]$ are computed by introducing the components of $\widehat{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$, respectively, in formulas (45) and (48).

2. Distinguished (pseudo) Riemannian spaces (DpR) are defined as manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^{a}_{i}, d–metric field and d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ (a distorsion of the Levi–Civita connection, or of the canonical d–connection) satisfying the condition $\mathbf{Q}_{\alpha\beta\gamma} = 0$.

3. Berwald (pseudo) Riemann spaces (pRB) are modeled if a N–connection structure is defined by a generic off–diagonal metric. The Levi–Civita connection is distorted to a Berwald d–connection ${}^{[B]}\mathbf{D} = [\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}] = [\widehat{L}^{i}{}_{jk}, \partial_b N^{a}_{k}, 0, \widehat{C}^{a}{}_{bc}]$, see (62). By definition, this space satisfies the metricity conditions in the h- and v–subspaces, $Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields because Q_{iab} and Q_{ajk} are not vanishing (see formulas (63)). Such nonmetricities vanish with respect to holonomic frames. The torsion is zero for the Levi–Civita connection but ${}^{[B]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]}]$ is not trivial.

4. Berwald (pseudo) Riemann spaces with prescribed d–torsion (pRBT) are defined by a more general class of d–connection ${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = [L^{i}{}_{jk}, \partial_b N^{a}_{k}, 0, C^{a}{}_{bc}]$ inducing prescribed values $\tau^{i}{}_{jk}$ and $\tau^{a}{}_{bc}$ in d–torsion ${}^{[BT]}\mathbf{T}^{\alpha}{}_{\beta\gamma} = [L^{i}{}_{[jk]}, +\tau^{i}{}_{jk}, 0, \Omega^{a}_{ij}, T^{a}{}_{bj}, C^{a}{}_{[bc]} + \tau^{a}{}_{bc}]$, see (65). The nontrivial components of nonmetricity d–fields are ${}^{[B\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left({}^{[B\tau]}Q_{cij},\ {}^{[B\tau]}Q_{iab}\right)$. Such components vanish with respect to holonomic frames.

5. Generalized Lagrange (pseudo) Riemannian spaces (pRGL) are modeled as distinguished Riemann spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generic off–diagonal metrics with associated N–connection inducing a tangent bundle structure. The d–metric $\mathbf{g}_{[a]}$ (112) and the d–connection ${}^{[a]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta} = \left({}^{[a]}L^{i}_{jk},\ {}^{[a]}C^{i}_{jc}\right)$ (111) are those for the usual Lagrange spaces (see Definition 4.3) but on a (pseudo) Riemann manifold with prescribed N–connection structure.

6. Lagrange (pseudo) Riemann spaces (pRL) are provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (101) inducing the canonical N–connection structure ${}^{[cL]}\mathbf{N} = \{ {}^{[cL]}N^i_j\}$ (102) for a Lagrange space $\mathbf{L}^n = (V^n, g_{ij}(x,y))$ and, for instance, provided with a canonical d–connection structure ${}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha} = {}^{[b]}\mathbf{\Gamma}^{\gamma}{}_{\alpha\beta}\vartheta^{\beta}$ satisfying metricity conditions for the d–metric defined by $g^{[L]}_{ij}(x,y)$. There is an alternative construction with Levi–Civita connection.

7. Finsler (pseudo) Riemann (FpR) are defined by a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) constructed from a Finsler metric $F\left(x^i, y^j\right)$. It is induced the canonical N–connection structure ${}^{[F]}\mathbf{N} = \{ {}^{[F]}N^i_j\}$ (86) as in the Finsler space $\mathbf{F}^n = (V^n, F(x,y))$ with ${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}{}_{\beta\gamma}$ being the canonical Finsler d–connection (88).

A.11 Teleparallel spaces

Teleparallel spaces were considered in sections 4.1.3 and 4.2.4. Here we classify what type of locally isotropic and anisotropic structures can be modeled in by anholonomic transforms of (pseudo) Riemannian spaces to teleparallel ones. The anholonomic frame structures are with associated N–connection with the components defined by the off–diagonal metric coefficients.

1. Teleparallel spaces (in brief, T) are usual ones with vanishing curvature, modeled on manifolds V^{n+m} of necessary smoothly class provided, for instance, with the Weitzenbock connection ${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$(82) which can be transformed in a d–connection one with respect to N–adapted frames. In general, such geometries are characterized by torsion ${}^{[W]}T^{\alpha}{}_{\beta\gamma}$ constrained to the condition to result in zero d–curvatures. The simplest theories are with vanishing nonmetricity.

2. Distinguished teleparallel spaces (DT) are manifolds $\mathbf{V}^{n+m}$ provided with N–connection structure N^a_i, d–metric field (33) and arbitrary d–connection $\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ with vanishing d–curvatures. The geometric constructions are stated by the Weitzenbock d–connection ${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ with distorsions without nonmetricity d–fields preserving the condition of zero values for d–curvatures.

3. Teleparallel Berwald spaces (TB) are defined by distorsions of the Weitzenbock connection on a manifold V^{n+m} to any Berwald like strucutre, ${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma} + \mathbf{Z}^{\alpha}{}_{\beta\gamma}$ satisfying the condition that the curvature is zero. All constructions with effective off–diagonal metrics can be adapted to the N–connection and considered for d–objects. Such spaces satisfy the metricity conditions in the h- and v–subspaces,

$Q_{ijk} = 0$ and $Q_{abc} = 0$, but, in general, there are nontrivial nonmetricity d–fields, Q_{iab} and Q_{ajk}.

4. Teleparallel Berwald spaces with prescribed torsion (TBT) are defined by a more general class of distorsions resulting in the Weitzenbock d–connection,

$$^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} + \mathbf{Z}^{\alpha}_{\ \beta\gamma},$$

having prescribed values $\tau^{i}_{\ jk}$ and $\tau^{a}_{\ bc}$ in d–torsion

$$^{[WB]}\mathbf{T}^{\alpha}_{\ \beta\gamma} = [L^{i}_{\ [jk]}, +\tau^{i}_{\ jk}, 0, \Omega^{a}_{ij}, T^{a}_{\ bj}, C^{a}_{\ [bc]} + \tau^{a}_{\ bc}]$$

and characterized by the condition $^{[WB\tau]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ with certain nontrivial nonmetricity d–fields, $^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = \left(^{[WB\tau]}Q_{cij},\ ^{[WB\tau]}Q_{iab}\right).$

5. Teleparallel generalized Lagrange spaces (TGL) are modeled as Riemann–Cartan spaces of odd–dimension, $\mathbf{V}^{n+n}$, provided with generalized Lagrange d–metric and associated N–connection inducing a tangent bundle structure with zero d–curvature. The Weitzenblock–Lagrange d–connection $^{[Wa]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = (\ ^{[Wa]}L^{i}_{jk},\ ^{[Wa]}C^{i}_{jc})$, where $^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} + \mathbf{Z}^{\alpha}_{\ \beta\gamma}$, are defined by a d–metric $\mathbf{g}_{[a]}$ (112) with $\mathbf{Z}^{\alpha}_{\ \ \beta}$ inducing zero nonmetricity d–fields, $^{[a]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ and zero d–curvature, $^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$.

6. Teleparallel Lagrange spaces (TL, see section 4.1) are Riemann–Cartan spaces $\mathbf{V}^{n+n}$ provided with a Lagrange quadratic form $g^{[L]}_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}$ (101) inducing the canonical N–connection structure $^{[cL]}\mathbf{N} = \{\ ^{[cL]}N^{i}_{j}\}$ (102) for a Lagrange space $\mathbf{L}^{n} = (V^{n}, g_{ij}(x,y))$ but with vanishing d–curvature. The d–connection structure $^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ (of Weitzenblock–Lagrange type) is the generated as a distortion by the Weitzenbock d–torsion, $^{[W]}\mathbf{T}_{\beta}$, but zero nonmetricity d–fields, $^{[WL]}\mathbf{Q}_{\beta\gamma\alpha} = 0$, when $^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} + \mathbf{Z}^{\alpha}_{\ \beta\gamma}$.

7. Teleparallel Finsler spaces (TF) are Riemann–Cartan manifolds $\mathbf{V}^{n+n}$ defined by a quadratic form $g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ (85) and a Finsler metric $F\left(x^i, y^j\right)$. They are provided with a canonical N–connection structure $^{[F]}\mathbf{N} = \{\ ^{[F]}N^{i}_{j}\}$ (86) as in the Finsler space $\mathbf{F}^{n} = (V^{n}, F\left(x,y\right))$ but with a Finsler–Weitzenbock d–connection structure $^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$, respective d–torsion, $^{[WF]}\mathbf{T}_{\beta}$, and vanishing nonmetricity, $^{[WF]}\mathbf{Q}_{\beta\gamma\tau} = 0$, d–fields, $^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta} = \mathbf{\Gamma}^{\alpha}_{\bigtriangledown\ \beta\gamma} + \hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma} + \mathbf{Z}^{\alpha}_{\ \beta\gamma}$,where $\hat{\mathbf{Z}}^{\alpha}_{\ \beta\gamma}$ contains distorsions from the canonical Finsler d–connection (88).

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^{\alpha}_{\beta\gamma}$ $T^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma}$
2. DMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ $\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma}$
3. BA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[B]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BAT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLA	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[a]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma}$
6. LA	$\dim i = \dim a$ ${}^{[cL]}N_j^i,\ {}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[b]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[b]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[b]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[b]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[L]}$
7. FA	$\dim i = \dim a$ ${}^{[F]}N_j^i;\ {}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[f]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[f]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[f]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma} = -\ {}^{[f]}\mathbf{D}_{\alpha}\mathbf{g}_{\beta\gamma}^{[F]}$

Table 1: Generalized Lagrange–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\beta\gamma}$ $\check{T}^{\alpha}{}_{\beta\gamma}$	$\check{R}^{\alpha}{}_{\beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	$\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BDA	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [0, \check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}, 0]$
4. BDAT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}^{[BT]}_{\alpha\beta\gamma} = [0, \check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}, 0]$
5. GHA	$\dim i = \dim a.$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{[a]} = [g_{ij}, \check{h}^{ij}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
6. HA	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia}$, ${}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[H]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[L]}_{\beta\gamma}$
7. CA	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia}$; ${}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[C]}\mathbf{D}_{\alpha}\mathbf{g}^{[C]}_{\beta\gamma}$

Table 2: Generalized Hamilton–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TMA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ ${}^{[W]}T^{\alpha}_{\ \beta\gamma}$	${}^{[W]}R^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[W]}Q_{\alpha\beta\gamma}$
2. DTMA	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma}$
3. TBA	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[WB]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. TBAT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[WB\tau]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. TGLA	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[Wa]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma}$
6. TLA	$\dim i = \dim a$ ${}^{[cL]}N_j^i$, ${}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}^{[L]}_{\alpha\beta}$	${}^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WL]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WL]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_{\alpha}\mathbf{g}^{[L]}_{\beta\gamma}$
7. TFA	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}^{[F]}_{\alpha\beta}$	${}^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WF]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WF]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} = -\mathbf{D}_{\alpha}\mathbf{g}^{[F]}_{\beta\gamma}$

Table 3: Teleparallel Lagrange–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TMDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[W]}\check{\Gamma}^{\alpha}_{\beta\gamma}$ ${}^{[W]}\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\check{Q}_{\alpha\beta\gamma}$
2. DTMDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. TBDA	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
4. TDBAT	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ off.d.m. $g_{\alpha\beta}$, $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB\tau]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WB\tau]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
5. TDGHA	$\dim i = \dim a.$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ij}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
6. TDHA	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia},\ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[WH]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[WH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[H]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[L]}_{\beta\gamma}$
7. TDCA	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia};\ \ {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[CW]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[CW]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[CW]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[CW]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = -\ {}^{[CW]}\check{\mathbf{D}}_{\alpha}\check{\mathbf{g}}^{[C]}_{\beta\gamma}$

Table 4: Teleparallel Hamilton–affine spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MAVB	N_i^a, Ω_{ij}^a,off.d.m vect.bundle $g_{\alpha\beta}$, total space $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^\alpha_{\beta\gamma}$, total space $T^\alpha_{\ \beta\gamma}$	$R^\alpha_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma} = 0$; $Q_{\alpha\beta\gamma} \neq 0$ for MA str.
2. DMAVB	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^\alpha_{\beta\gamma}$ $\mathbf{T}^\alpha_{\ \beta\gamma}$	$\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma} = 0$; $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$ for DMA str.
3. BMATB	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[B]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[B]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BMATBT	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[BT]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[BT]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLMAB	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{[gL]} = [g_{ij}, h_{kl}]$	$\widehat{\mathbf{\Gamma}}^\alpha_{\ \beta\gamma}, \mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ $\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}, \mathbf{T}^\alpha_{\ \beta\gamma}$	$\widehat{\mathbf{R}}^\alpha_{\ \beta\gamma\tau}, \mathbf{R}^\alpha_{\ \beta\gamma\tau}$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
6. LMAB	$\dim i = \dim a$ ${}^{[L]}N_i^a$, ${}^{[L]}\Omega_{ij}^a$ d–metr.$\mathbf{g}^{[L]}_{\alpha\beta} =$ $[g^{[L]}_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i \partial y^j}]$	${}^{[L]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[b]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[L]}\mathbf{T}^\alpha_{\ \beta\gamma}$, ${}^{[b]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[L]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[b]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[L]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ ${}^{[b]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$
7. FMAB	$\dim i = \dim a$ ${}^{[F]}N_j^i$; ${}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}^{[F]}_{\alpha\beta} =$ $[g^{[F]}_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}]$	${}^{[F]}\widehat{\mathbf{\Gamma}}^\alpha_{\ \beta\gamma}$ ${}^{[f]}\mathbf{\Gamma}^\alpha_{\ \beta\gamma}$ ${}^{[F]}\widehat{\mathbf{T}}^\alpha_{\ \beta\gamma}$ ${}^{[f]}\mathbf{T}^\alpha_{\ \beta\gamma}$	${}^{[F]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[f]}\mathbf{R}^\alpha_{\ \beta\gamma\tau}$ ${}^{[F]}\mathbf{Q}_{\alpha\beta\gamma} = 0$ ${}^{[f]}\mathbf{Q}_{\alpha\beta\gamma} \neq 0$

Table 5: Generalized Finsler–Lagrange spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ total space off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\beta\gamma}$ $\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ $\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BMADB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
4. BMADBT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}^{\ ab}, \check{Q}^{a}_{\ jk}]$
5. GMAHB	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, {}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, {}^{[GH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, {}^{[GH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. MAHB	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia},\ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}, \check{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$
7. MACB	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia};\ \ {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}, \check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$

Table 6: Generalized Hamilton–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. MADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ total space off.d.m. $\check{g}_{\alpha\beta}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\Gamma}^{\alpha}_{\beta\gamma}$ $\check{T}^{\alpha}_{\beta\gamma}$	$\check{R}^{\alpha}{}_{\beta\gamma\tau}$ $\check{Q}_{\alpha\beta\gamma} = -\check{D}_{\alpha}\check{g}_{\beta\gamma}$
2. DMADVB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	$\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ $\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	$\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ $\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. BMADB	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[B]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[B]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[B]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[B]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}]$
4. BMADBT	$\check{N}_{ai}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[BT]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[BT]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[BT]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[BT]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{i}{}^{ab}, \check{Q}^{a}{}_{jk}]$
5. GMAHB	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[a]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}, {}^{[GH]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ ${}^{[a]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}, {}^{[GH]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[a]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}, {}^{[GH]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[a]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. MAHB	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia},\ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[H]}\check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}, \check{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta}$ ${}^{[H]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}, \check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[H]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[H]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ $\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$
7. MACB	$\dim i = \dim a$ ${}^{[C]}\check{N}_{ia};\quad {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[C]}\check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}, \check{\mathbf{\Gamma}}^{\alpha}_{\beta\gamma}$ ${}^{[C]}\check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}, \check{\mathbf{T}}^{\alpha}{}_{\beta\gamma}$	${}^{[C]}\check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}, \check{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$ ${}^{[C]}\check{\mathbf{Q}}_{\alpha\beta\gamma} \neq 0$

Table 7: Teleparallel Finsler–Lagrange spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. TDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{g}_{\alpha\beta}$, d. vect. b. $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$,	${}^{[W]}\check{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[W]}\check{T}^{\alpha}_{\ \beta\gamma}$	$\check{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ $\check{Q}_{\alpha\beta\gamma}$
2. DTDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma}$
3. TBDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{iab}, \check{Q}_{ajk}]$
4. TBDVB	$\check{N}_{ai}, \check{\Omega}_{iaj} = \delta_{[i}\check{N}_{j]a}$ $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ab}]$	${}^{[WB\tau]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB\tau]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = [\check{Q}_{iab}, \check{Q}_{ajk}]$
5. TGH	$\dim i = \dim a$ $\check{N}_{ia}, \check{\Omega}_{iaj}$, d. tan. b. $\check{\mathbf{g}}_{\alpha\beta} = [g_{ij}, \check{h}^{ij}]$	${}^{[Wa]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[Wa]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
6. TH	$\dim i = \dim a$ ${}^{[H]}\check{N}_{ia},\ {}^{[H]}\check{\Omega}_{iaj}$ d–metr.$\check{\mathbf{g}}^{[H]}_{\alpha\beta}$	${}^{[WH]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[WH]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[WH]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WH]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
7. TC	$\dim i = \dim a.$ ${}^{[C]}\check{N}_{ia};\ \ {}^{[C]}\check{\Omega}_{iaj}$ d–metr. $\check{\mathbf{g}}^{[C]}_{\alpha\beta}$	${}^{[CW]}\check{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[CW]}\check{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[CW]}\check{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[CW]}\check{\mathbf{Q}}_{\alpha\beta\gamma} = 0$

Table 8: Teleparallel Hamilton–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. RC	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\Gamma^{\alpha}_{\beta\gamma}$ $T^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \beta\gamma\tau}$ $Q_{\alpha\beta\gamma} = 0$
2. DRC	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	$\mathbf{\Gamma}^{\alpha}_{\beta\gamma}$ $\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $\mathbf{Q}_{\alpha\beta\gamma} = 0$
3. BRC	N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[B]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[B]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[B]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. BRCT	N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[BT]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. GLRC	$\dim i = \dim a$ N_i^a, Ω_{ij}^a off diag. $g_{\alpha\beta}$, $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[a]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[a]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[a]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[a]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
6. LRC	$\dim i = \dim a$ ${}^{[cL]}N_j^i,\ {}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	$\hat{\mathbf{\Gamma}}^{\gamma}_{\ \alpha\beta}$ $\hat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$\hat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$
7. FRC	$\dim i = \dim a.$ ${}^{[F]}N_j^i;\ {}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}$ ${}^{[F]}\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	${}^{[F]}\hat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ ${}^{[F]}\hat{\mathbf{Q}}_{\alpha\beta\gamma} = 0$

Table 9: Distinguished Riemann–Cartan spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. pR	N_i^a, off-d.metr. $\Omega^a_{ij}=0,\neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta}=$ $[g_{ij},h_{ab}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\widehat{\mathbf{D}}=[\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ $\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}\neq 0$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma}=0$
2. DpR	N_i^a, off-d.metr. $\Omega^a_{ij}=0,\neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta}=$ $[g_{ij},h_{ab}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\mathbf{D}=[\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ $\mathbf{T}^{\alpha}_{\ \beta\gamma}\neq 0$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ $\mathbf{Q}_{\alpha\beta\gamma}=0$
3. pRB	N_i^a, off-d.metr. $\Omega^a_{ij}=0,\neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta}=$ $[g_{ij},h_{ab}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[B]}\mathbf{D}=[\ {}^{[B]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ ${}^{[B]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[B]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ ${}^{[B]}\mathbf{Q}_{\alpha\beta\gamma}\neq 0$
4. pRBT	N_i^a, off-d.metr. $\Omega^a_{ij}=0,\neq 0$ $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta}=$ $[g_{ij},h_{ab}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[BT]}\mathbf{D}=[{}^{[BT]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ ${}^{[BT]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[BT]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ ${}^{[BT]}\mathbf{Q}_{\alpha\beta\gamma}\neq 0$
5. pRGL	$N_i^a;\dim i=\dim a$ $\Omega^a_{ij}=0,\neq 0$ $\mathbf{g}_{\alpha\beta}=$ $[g_{ij},h_{ab}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ $\widehat{\mathbf{D}}=[\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ $\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ $\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ $\widehat{\mathbf{Q}}_{\alpha\beta\gamma}=0$
6. pRL	${}^{[L]}N_i^a;\dim i=\dim a$ ${}^{[L]}\Omega^a_{ij}=0,\neq 0$ $\mathbf{g}^{[L]}_{\alpha\beta}=[g^{[L]}_{ij},g^{[L]}_{ij}]$ $[g^{[L]}_{ij}=\frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[L]}\mathbf{D}=[\ {}^{[L]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ ${}^{[L]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[L]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ ${}^{[L]}\mathbf{Q}_{\alpha\beta\gamma}=0$
7. pRF	${}^{[F]}N_i^a;\dim i=\dim a$ ${}^{[F]}\Omega^a_{ij}=0,\neq 0$ $\mathbf{g}^{[F]}_{\alpha\beta}=[g^{[F]}_{ij}]$ $[g^{[F]}_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\partial y^j}]$	$\bigtriangledown=[\Gamma^{\alpha}_{\ \bigtriangledown\beta\gamma}]$ ${}^{[F]}\widehat{\mathbf{D}}=[\ {}^{[F]}\widehat{\mathbf{\Gamma}}^{\alpha}_{\ \beta\gamma}]$ $T^{\alpha}_{\bigtriangledown\ \beta\gamma}=0$ ${}^{[F]}\widehat{\mathbf{T}}^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \bigtriangledown\beta\gamma\tau}$ ${}^{[F]}\widehat{\mathbf{R}}^{\alpha}_{\ \beta\gamma\tau}$ $Q^{\bigtriangledown}_{\alpha\beta\gamma}=0$ ${}^{[F]}\widehat{\mathbf{Q}}_{\alpha\beta\gamma}=0$

Table 10: Distinguished (pseudo) Riemannian spaces

Space	N–connection/ N–curvature metric/ d–metric	(d–)connection/ (d-)torsion	(d-)curvature/ (d–)nonmetricity
1. T	N_i^a, Ω_{ij}^a off.d.m. $g_{\alpha\beta}$, $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[W]}\Gamma^{\alpha}_{\beta\gamma}$ ${}^{[W]}T^{\alpha}_{\ \beta\gamma}$	$R^{\alpha}_{\ \beta\gamma\tau} = 0$ $Q_{\alpha\beta\gamma} = 0$
2. DT	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[Wa]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
3. TB	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
4. TBT	N_i^a, Ω_{ij}^a $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$	${}^{[WB\tau]}\mathbf{\Gamma}^{\alpha}_{\ \beta\gamma}$ ${}^{[WB\tau]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WB\tau]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WB\tau]}\mathbf{Q}_{\alpha\beta\gamma} = [Q_{iab}, Q_{ajk}]$
5. TGL	$\dim i = \dim a$ N_i^a, Ω_{ij}^a $\mathbf{g}_{[a]} = [g_{ij}, h_{kl}]$	${}^{[Wa]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[Wa]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[Wa]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[Wa]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
6. TL	$\dim i = \dim a$ ${}^{[cL]}N_j^i,\ {}^{[cL]}\Omega_{ij}^a$ d–metr.$\mathbf{g}_{\alpha\beta}^{[L]}$	${}^{[WL]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WL]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WL]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WL]}\mathbf{Q}_{\alpha\beta\gamma} = 0$
7. TF	$\dim i = \dim a$ ${}^{[F]}N_j^i;\ {}^{[F]}\Omega_{ij}^k$ d–metr.$\mathbf{g}_{\alpha\beta}^{[F]}$	${}^{[WF]}\mathbf{\Gamma}^{\gamma}_{\ \alpha\beta}$ ${}^{[WF]}\mathbf{T}^{\alpha}_{\ \beta\gamma}$	${}^{[WF]}\mathbf{R}^{\alpha}_{\ \beta\gamma\tau} = 0$ ${}^{[WF]}\mathbf{Q}_{\alpha\beta\gamma} = 0$

Table 11: Teleparallel spaces